

Virtual Calculus Tutor

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Movie:  2.1 Motivating the Idea of Slope of a Curved Graph

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Movie:  **2.2 Introduction to the Limit Concept**

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Example 4: $f(x) = \frac{x^2 - 9}{x - 3}$ for $x \neq 3$

Example 5: $f(x) = x + 3$ for all x

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Movie:  **2.3 Properties of Limits**

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Document: 2.4 Trigonometric Limits

Movie:  2.4 Trigonometric Limits

2.4.1 Radian Measure and Area of a Circular Sector

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 Exercise 11: $\lim_{\theta \rightarrow 0} \frac{\sqrt[3]{\cos 3\theta} - \sqrt[3]{\cos 5\theta}}{\theta^2}$
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 Exercise 13: $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$

Document: 2.5 Continuity

Movie:  2.5 Continuity

2.5.1 Introducing the Concept of Continuity

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 Definition of Continuity of a Function f at a Number x

2.5.2 Some Examples to Illustrate the Idea of a of Continous Function

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 Example 3: $f(t) = t + 3$ for $t \neq 3$

Example 4: $f(t) = \frac{t^2 - 9}{t - 3}$ when $t - 3 \neq 0$

Example 5: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 6: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 2 & \text{if } t = 3 \end{cases}$

Example 7: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t < 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 8: $f(t) = \begin{cases} t + 3 & \text{if } t < 3 \\ 6 & \text{if } t = 3 \\ 2 - t & \text{if } t > 3 \end{cases}$

2.5.3 Properties of Continuous Functions

Preliminary Comment

The Bolzano Intermediate Value Theorem

Introduction to the Bolzano Intermediate Value Theorem

Statement of the Bolzano Intermediate Value Theorem

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The Intermediate Value Property

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The Theorem on Existence of Maxima and Minima of Continuous Functions

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The Effect of a Missing Endpoint

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2.5.5 Exercises on the Properties of Continuous Functions

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Exercise 3: $f(x) = \begin{cases} x & \text{if } 0 < x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 4 \end{cases}$

Exercise 4: $f(x) = |x^2 - 4|$ for $0 \leq x \leq 5/2$

Exercise 5: $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1 + 4x - x^2 & \text{if } 1 \leq x \leq 4 \end{cases}$

Exercise 6: Existence of a solution of $5\sqrt[3]{x} + \sqrt{9-x} = 6$

Overview of Chapter 3: Derivatives

Document: 3.1 Introduction to Derivatives

Movie:



3.1 Introduction to Derivatives

3.1.1 Definition of a Derivative

Motivating the Definition Using Slopes

Definition of the Derivative of a Function

Alternative Form of the Definition of a Derivative

3.1.2 Some Examples of Derivatives

Example 1: Derivative of a constant

Example 2: $f(x) = mx + b$ for all x

Example 3: $f(x) = x^2$ for all x , find $f'(3)$

Example 4: $f(x) = x^2$ for all x , find $f'(x)$

Example 5: $f(x) = x^3$ for all x , find $f'(x)$

Example 6: $f(x) = x^7$ for all x , find $f'(x)$

3.1.3 The Power Rule

Introducing the Power Rule
The Power Rule for the Case $p = -5$
The Power Rule for the Case $p = 5/6$
The Power Rule for the Case $p = -4/7$
The Power Rule for Fractional Exponents
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3.1.4 Derivatives of Polynomials

Introducing the Idea of a Polynomial
Finding the Derivative of a Polynomial

3.1.5 The Leibniz Notation for Derivatives

Motivating the Leibniz Notation for Derivatives
Introducing the Leibniz Notation for Derivatives
The Power Rule in Leibniz Notation
Derivative of a Polynomial in Leibniz Notation

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Exercise 2: $\frac{d}{dx} \frac{5}{\sqrt[3]{x^7}}$

Exercise 3: $y = 8x^3 - 6x - 1$ tangent line problem

Exercise 4: $y = \frac{1}{\sqrt{x}}$ tangent line problem

Exercise 5: Tangent from $(-2, -21)$ to $y = x^2$

Exercise 6: Tangent from $(-3, 1)$ to $y = \frac{1}{x}$

Exercise 7: $f(x) = |x - 3|$ no derivative at 3

Exercise 8: $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$

Exercise 9: $x^2 \sin \frac{1}{x}$ derivative at 0?

Document: 3.2 Elementary Facts About Derivatives

Movie:



3.2 Elementary Facts About Derivatives

3.2.1 The Rules for Differentiation

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Stating the Sum Rule
Explaining the Sum Rule
The Difference Rule
Stating the Difference Rule
Explaining the Difference Rule
The Constant Multiple Rule
Stating the Constant Multiple Rule
Explaining the Constant Multiple Rule
The Product Rule
Stating the Product Rule
A Needed Fact About Limits
Explaining the Product Rule
The Quotient Rule
Stating the Quotient Rule
Explaining the Quotient Rule
An Optional Deeper Comment About the Proof of the Quotient Rule

3.2.2 Exercises on the Rules for Differentiation

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Exercise 2: Tangent line from $(4, 4)$ to $y = x + \frac{1}{x}$

Exercise 3: $\frac{d}{dx} \frac{x}{1+x^2}$

Exercise 4: Horizontal tangents to $y = \frac{x^2}{1+x^4}$

Exercise 5: $y = \frac{x}{x^2+4}$ tangent line problem

Exercise 6: $f(x) = (x-3)^2 g(x)$ tangent line problem

Exercise 7: $\frac{d}{dx} f(x)g(x)h(x)$ extended product rule

Exercise 8: $\frac{d}{dx} (f(x))^2 = 2f(x)f'(x)$

Exercise 9: Horizontal tangents to $y = (2 - 3x)^5(5 + 2x)^4$

3.2.3 Higher Order Derivatives

3.2.4 Exercises on Higher Order Derivatives

Exercise 1: $f(x) = x^7$ for each x , work out $f^{(n)}(x)$

Exercise 2: $f(x) = \sqrt{x}$ for each $x > 0$, work out $f^{(n)}(x)$

Exercise 3: $f(x) = \frac{1}{1+x^2}$ find $f''(x)$

Exercise 4: Expand $(1+x)^7$ using derivatives

Exercise 5: Expand $(1+x)^p$ using derivatives

Document: 3.3 Derivatives of the Trigonometric Functions

Movie:  3.3 Derivatives of the Trigonometric Functions

3.3.1 Derivatives of the Functions sin and cos

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The Derivative of cos

3.3.2 Derivatives of the Other Trigonometric Functions

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Finding the Derivative of tan Directly from the Definition
The Derivative of cot
Finding the Derivative of cot Directly from the Definition
The Derivative of sec
Finding the Derivative of sec Directly from the Definition
The Derivative of csc
Finding the Derivative of csc Directly from the Definition
Summary of the Trigonometric Derivatives

3.3.3 Exercises on Derivatives of the Trigonometric Functions

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Exercise 2: $\frac{d}{dx} x^2 \sin x \cos x$

Exercise 3: $\frac{d}{dx} \frac{x \sin x}{1+x^2}$

Exercise 4: Horizontal tangents to $y = 2 \cos^2 x + 2 \cos x - 1$

Exercise 5: $\frac{d}{dx} ((f(x) - \sin x)^2 + (g(x) - \cos x)^2)$

Document: 3.4 Derivative of a Composition

Movie:  3.4 Derivative of a Composition

3.4.1 Composition of Functions

3.4.2 Some Examples of Compositions

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Example 2: $f(x) = 1 + x^2$ for every number x and $g(u) = u^{100}$ for every number u

Example 3: $f(x) = 2^x$ for every number x and $g(u) = \log_2 u$ for $u > 0$

Example 4: $f(x) = \frac{x-2}{1-2x}$ whenever $x \neq \frac{1}{2}$ and $g(u) = \frac{u-3}{1-3u}$ for $u \neq \frac{1}{3}$


3.4.3 Statement of the Composition Rule

3.4.4 Some Examples to Illustrate the Composition Rule

Example 1: $\frac{d}{dx} (1+x^2)^{100}$

Example 2: $\frac{d}{dx} \sin(1+x^2)$

Example 3: $\frac{d}{dx} \sqrt{\sin x}$

 **3.4.5 Motivating the Composition Rule**

 **3.4.6 Using Leibniz Notation in the Composition Rule**

 **3.4.7 A Return to the Earlier Examples on the Composition**

Example 1: $\frac{d}{dx} (1 + x^2)^{100}$

Example 2: $\frac{d}{dx} \sin(1 + x^2)$

Example 3: $\frac{d}{dx} \sqrt{\sin x}$

 **3.4.8 Some Assorted Exercises on Derivatives**

Exercise 1: $\frac{d}{dx} \sqrt{\sin(1 + x^2)}$

Exercise 2: $\frac{d}{dx} (\sin x + \cos x)^{100}$

Exercise 3: $\frac{d}{dx} \sqrt{\sin \sqrt{x}}$

Exercise 4: $\frac{d}{dx} (\sin x + x \cos(x^3))^{100}$

Exercise 5: $\frac{d}{dx} \frac{\sqrt{\sin(x^3)}}{\sqrt[3]{\cos(x^2)}}$

Exercise 6: Tangent to $y = \tan x$ at $x = \pi/4$

Exercise 7: Tangent to $y = \sqrt{13 - x^2}$ at $(5, 1)$


Exercise 8: Finding the Angle Between Two Graphs


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Note on the Final Two Exercises

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 **Document: 3.5 Inverse Functions**

Movie:  **3.5 Inverse Functions**

 **3.5.1 Domain and Range of a Function**

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Example 2 on Domain and Range

Example 3 on Domain and Range

Example 4 on Domain and Range

 **3.5.2 Inverse Function of a One-One Function**

One-One Functions

Example 1 of a One-One Function


Example 2 of a One-One Function

Inverse of a One-One Function

Example 1 on Inverse Functions

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Introducing the Derivative of an Inverse Function

Example 1 of the Derivative of an Inverse Function

Example 2 of the Derivative of an Inverse Function

 **Document: 3.6 Derivatives of Exponential and Logarithmic Functions**

Movie:  **3.6 Derivatives of Exponential and Logarithmic Functions**

 **3.6.1 The Key to the Differentiation of an Exponential Function**

 **3.6.2 Approximate Differentiation an Exponential Function with a Computer Algebra System**

Choosing a Computer Algebra System
 Setting up Scientific Notebook
 Approximate Evaluation of $\frac{d}{dx} 2^x$
 Approximate Evaluation of $\frac{d}{dx} 3^x$

3.6.2 Approximate Differentiation an Exponential Function with a Computer Algebra System Interactive Form

Choosing a Computer Algebra System
 Setting up Scientific Notebook
 Approximate Evaluation of $\frac{d}{dx} 2^x$
 Approximate Evaluation of $\frac{d}{dx} 3^x$

3.6.3 Adjusting the Base of an Exponential Function: The Number e

Preliminary Note
 Our Objective: To Obtain $\frac{d}{dx} a^x = 1a^x$
 Adjusting the Base Numerically
 Adjusting the Base Geometrically: Animation Method
 Adjusting the Base Geometrically: Zooming Method
 Comparing the Graphs $y = a^x$ and $y = \frac{d}{dx} a^x$
 The Function exp

3.6.3 Adjusting the Base of an Exponential Function: The Number e Interactive Form

Preliminary Note
 Our Objective: To Obtain $\frac{d}{dx} a^x = 1a^x$
 Adjusting the Base Numerically
 Adjusting the Base Geometrically: Animation Method
 Adjusting the Base Geometrically: Zooming Method
 Comparing the Graphs $y = a^x$ and $y = \frac{d}{dx} a^x$
 The Function exp

3.6.4 A More Precise Approach to the Number e

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 Some Examples Involving the Exponential Function Base e
 Finding $\frac{d}{dx} a^x$ for a General Base a
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 The Equation $\frac{d}{dx} \log|x| = \frac{1}{x}$
 Finding $\frac{d}{dx} \log_a x$ for a General Base a

3.6.5 Some Exercises on Derivatives of Exponential and Logarithmic Functions

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- Exercise 2: $\frac{d}{dx} \log(5x)$
- Exercise 3: $\frac{d}{dx} \log 5 = 0$
- Exercise 4: $f(x) = \log(1 + x^2)$
- Exercise 5: $\frac{d}{dx} \log|\sin x|$
- Exercise 6: $\frac{d}{dx} \log|\sec x|$
- Exercise 7: $\frac{d}{dx} \log|\sec x + \tan x|$
- Exercise 8: $\frac{d}{dx} \log|\csc x + \cot x|$
- Exercise 9: $\frac{d}{dx} (1 + x^2)^{\sin x}$
- Exercise 10: $\frac{d}{dx} \log_{(1+x^2)}(1 + x^2 + 2x^4)$
- Exercise 11: $\lim_{x \rightarrow 0} (1 + x)^{1/x}$
- Exercise 12: $\lim_{u \rightarrow \infty} (1 + \frac{1}{u})^u$

Document: 3.7 Inverse Trigonometric Functions

Movie:  **3.7 Inverse Trigonometric Functions**

3.7.1 The Function arccos

3.7.2 Some Examples to Illustrate the Function arccos

The Number $\arccos 0$

The Number $\arccos \frac{1}{2}$

The Number $\arccos \left(-\frac{1}{2}\right)$

The Numbers $\arccos \frac{1}{\sqrt{2}}$ and $\arccos \left(-\frac{1}{\sqrt{2}}\right)$

The Numbers $\arccos \left(\frac{\sqrt{3}}{2}\right)$ and $\arccos \left(-\frac{\sqrt{3}}{2}\right)$

The Numbers $\arccos(.37)$ and $\arccos(-.37)$

3.7.3 Some Properties of the Function arccos

Working Out $\cos(\arccos x)$, $\sin(\arccos x)$, and $\tan(\arccos x)$

The Derivative of the Function arccos

The Graph of the Function arccos

3.7.4 The Function arcsin

3.7.5 Some Examples to Illustrate the Function arcsin

The Numbers $\arcsin 1$ and $\arcsin(-1)$

The Numbers $\arcsin \frac{1}{2}$ and $\arcsin \left(-\frac{1}{2}\right)$

The Numbers $\arcsin \frac{1}{\sqrt{2}}$ and $\arcsin \left(-\frac{1}{\sqrt{2}}\right)$

3.7.6 Some Properties of the Function arcsin

Working Out $\sin(\arcsin x)$, $\cos(\arcsin x)$, and $\tan(\arcsin x)$

The Derivative of the Function arcsin

The Graph of the Function arcsin

3.7.7 The Function arctan

3.7.8 Some Examples to Illustrate the Function arctan

The Number $\arctan 0$

The Numbers $\arctan 1$ and $\arctan(-1)$

The Numbers $\arctan \sqrt{3}$ and $\arctan(-\sqrt{3})$

The Numbers $\arctan \frac{1}{\sqrt{3}}$ and $\arctan \left(-\frac{1}{\sqrt{3}}\right)$

The Limits of \arctan at ∞ and at $-\infty$

3.7.9 Some Properties of the Function arctan

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Derivative of the Function arctan

The Graph of the Function arctan

3.7.10 The Function arcsec

3.7.11 Some Examples to Illustrate the Function arcsec

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The Numbers $\operatorname{arcsec} 2$ and $\operatorname{arcsec}(-2)$

The Numbers $\operatorname{arcsec} \sqrt{2}$ and $\operatorname{arcsec}(-\sqrt{2})$

3.7.12 Some Properties of the Function arcsec

Working Out $\sec(\operatorname{arcsec} x)$, $\tan(\operatorname{arcsec} x)$, and $\sin(\operatorname{arcsec} x)$

The Derivative of the Function arcsec

The Graph of the Function arcsec

3.7.13 Exercises on Inverse Trigonometric Functions

Exercise 1: $\arctan(\sqrt{2} - 1) = \frac{\pi}{8}$

Exercise 2: $\arctan(2 - \sqrt{3}) = \frac{\pi}{12}$

Exercise 3: $\cos(2 \arcsin u) + \cos(2 \arccos u) = 0$

Exercise 4: $\arccos(\cos \theta) = \theta?$

Exercise 5: $\cos(3 \arccos x) = 4x^3 - 3x$

Exercise 6: $\sin(4 \arccos x) = 4x(2x^2 - 1)\sqrt{1 - x^2}$

Exercise 7: $\tan(2 \arctan x)$ defined?

Exercise 8: $\arctan x + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$ for $x < 0$

Exercise 9: $\arcsin(-x) = -\arcsin x$

Exercise 10: $\arccos(-x) = \pi - \arccos x$

Exercise 11: $\arctan\left(\frac{1 - \cos \theta}{\sin \theta}\right) + \arctan(\cot \theta) = \frac{\pi - \theta}{2}$

Document: 3.8 Implicit Functions

Movie:



3.8 Implicit Functions

3.8.1 Implicit 2D Graphs

Example 1: $x^2 + y^2 = 25$

Example 2: $x^2y - y^2 + xy^3 = 5$

Example 3: $(x^2 + y^2)^2 = x^2 - y^2$

Example 4: $x^3 + y^3 - 3xy = 0$

Example 5: $x^5 + y^5 - 3x^2y = 0$

Example 6: $x \sin(x^2 + y^2) + y = 0$

3.8.2 The Implicit Function Theorem

3.8.3 Some Exercises on Implicit Functions

Exercise 1: Tangent to $x^2 + y^2 = 25$ at $(3, 4)$

Exercise 2: Slope of $x^2y - y^2 + xy^3 = 5$ at a general point (x, y)

Exercise 3: Tangent to $x^2y - y^2 + xy^3 = 5$ at $(2, 1)$

Exercise 4: Slope of $(x^2 + y^2)^2 = x^2 - y^2$ at a general point (x, y)

Exercise 5: Horizontal and vertical tangents to $x^3 + y^3 - 3xy = 0$

Exercise 6: Horizontal and vertical tangents to $x^5 + y^5 - 3x^2y = 0$

Exercise 7: Slope of $x \sin(x^2 + y^2) + y = 0$ at a general point (x, y)

Document: 3.9 Hyperbolic Functions

Movie:



3.9 Hyperbolic Functions

3.9.1 Introduction to Hyperbolic Functions

Some Preliminary Comments

The Definitions of the Hyperbolic Functions

3.9.2 Arithmetical Properties of the Hyperbolic Functions

Behaviour of the Hyperbolic Functions at 0

"Pythagorean Identities" for the Hyperbolic Functions

Replacing x by $-x$ in the Hyperbolic Functions

Hyperbolic Function Values at a Sum or Difference

Analogues for the Hyperbolic Functions of the Trigonometric Double and Triple Angle Identities

3.9.3 Derivatives of the Hyperbolic Functions

The Equation $\frac{d}{dx} \sinh x = \cosh x$

The Equation $\frac{d}{dx} \cosh x = \sinh x$

The Equation $\frac{d}{dx} \tanh x = \text{sech}^2 x$

The Equation $\frac{d}{dx} \text{sech } x = -\text{sech } x \tanh x$

3.9.4 Inverse Functions of the Hyperbolic Functions

The Function $\operatorname{arcsinh}$

Finding $\frac{d}{dx} \operatorname{arcsinh} x$

The Function $\operatorname{arccosh}$

Finding $\frac{d}{dx} \operatorname{arccosh} x$

The Function $\operatorname{arctanh}$

Finding $\frac{d}{dx} \operatorname{arctanh} x$

The Function $\operatorname{arcsech}$

Finding $\frac{d}{dx} \operatorname{arcsech} x$

3.9.5 Some Derivatives that Involve the Hyperbolic Functions

Example 1: $\frac{d}{dx} \arctan(\sinh x)$

Example 2: $\frac{d}{dx} \arctan(e^x)$

Example 3: $\frac{d}{dx} \arcsin(\operatorname{sech} x)$

Example 4: $\frac{d}{dx} \log(\cosh x)$

Example 5: $\frac{d}{dx} \log(\sinh x)$

Example 6: $\frac{d}{dx} \operatorname{arcsec}(\cosh x)$

Example 7: $\frac{d}{dx} \operatorname{arccos}(\operatorname{sech} x)$

Example 8: $\frac{d}{dx} \operatorname{arccosh}(\sec x)$

Example 9: $\frac{d}{dx} \operatorname{arctanh}(\sin x)$

Overview of Chapter 4: Applications of the Derivative

Document: 4.1 Monotone Functions

Movie:  4.1 Monotone Functions

4.1.1 The Graph of a Function with a Positive Derivative

The Positive Derivative Principle
Looking at the Positive Derivative Principle Intuitively
A Note of Caution

4.1.2 Increasing and Decreasing Functions

Strictly Increasing Functions
Increasing Functions
Strictly Decreasing Functions
Decreasing Functions
Monotone Functions

4.1.3 More General Version of the Positive Derivative Principle

4.1.4 Exercises on Monotone Functions

Exercise 1: $f(x) = x^2 - 4x - 5$ for all x

Exercise 2: $f(x) = x^3 - 3x^2$ for all x

Exercise 3: $f(x) = f(x) = |x^2 - 4x - 5|$ for all x

Exercise 4: $f(x) = |x^3 - 3x^2|$ for all x

Exercise 5: $f(x) = \left(\frac{\log x}{x}\right)^2$ for $x > 0$

4.1.5 An Application of the Positive Derivative Principle

The Inequality $e^x > 1$ when $x > 0$

The Inequality $e^x > 1 + x$ when $x > 0$

The Inequality $e^x > 1 + x + \frac{x^2}{2}$ when $x > 0$

The Inequality $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$ when $x > 0$

The General Case $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$ when $x > 0$

4.1.6 Working out Some Important Limits

The Limit $\lim_{x \rightarrow \infty} \frac{e^x}{x}$

The Limit $\lim_{x \rightarrow \infty} \frac{e^x}{x^5}$

The Limit $\lim_{x \rightarrow \infty} \frac{e^x}{x^n}$

The Limit $\lim_{x \rightarrow \infty} \frac{\log x}{x}$

The Limit $\lim_{x \rightarrow \infty} \frac{(\log x)^{1000000}}{x}$

The Limit $\lim_{x \rightarrow 0^+} x \log x$

The Limit $\lim_{x \rightarrow 0^+} x(\log x)^{1000000}$

Document: 4.2 Drawing Graphs of Functions

Movie:



4.2 Drawing Graphs of Functions

4.2.1 Maxima and Minima

Definition of Maxima and Minima

Definition of Local Maxima and Minima

4.2.2 Fermat's Theorem

Statement of Fermat's Theorem

Part 1: Positive derivative not at the right endpoint

Part 2: Negative derivative not at the right endpoint

Part 3: Positive derivative not at the left endpoint

Part 4: Negative derivative not at the left endpoint

Part 5: Conclusion

Using Fermat's Theorem

Critical Numbers of a Function

4.2.3 Some Examples to Illustrate Fermat's Theorem

Example 1: $f(x) = x^3$ for $-2 \leq x \leq 2$

Example 2: $f(x) = x^3$ for $-2 \leq x < 2$

Example 3: $f(x) = \begin{cases} \frac{x-1}{2} & \text{if } 0 \leq x \leq 3 \\ 2x-5 & \text{if } 3 \leq x \leq 4 \end{cases}$

4.2.4 Exercises on Graphs of Functions

Exercise 1: $f(x) = x^2 - 4x - 5$ for $-2 \leq x \leq 6$

Exercise 2: $f(x) = x^2 - 4x - 5$ for $3 \leq x \leq 6$

Exercise 3: $f(x) = |x^2 - 4x - 5|$ for $-2 \leq x \leq 6$

Exercise 4: $f(x) = x^3 - 3x^2$ for $-1 \leq x \leq 4$

Exercise 5: $f(x) = \frac{x^2}{1+x^2}$ for all x

Exercise 6: $f(x) = xe^{-x}$ for $x \geq -1$

Exercise 7: $f(x) = xe^{-x^2}$ for all x

Exercise 8: $f(x) = x^2 e^{-x^2}$ for all x

Exercise 9: $f(x) = 3 \sin^4 x - 2 \sin^3 x$ for $0 \leq x \leq 2\pi$

Exercise 10: $f(x) = x(\log x)^2$ for $0 < x \leq 2$

Exercise 11: $f(x) = x^{2/3}(6-x)^{1/3}$ for $-1 \leq x \leq 7$

4.2.5 Concavity of Graphs

The Graph of a Function with a Positive Second Derivative

The Graph of a Function with a Negative Second Derivative

Points of Inflection

4.2.6 Exercises on Concavity

Exercise 1: $f(x) = x^3 - 3x^2$ for all x

Exercise 2: $f(x) = \frac{x^2}{1+x^2}$ for all x

Exercise 3: $f(x) = xe^{-x}$ for all x

- Exercise 4: $f(x) = xe^{-x^2}$ for all x
 Exercise 5: $f(x) = x^2e^{-x^2}$ for all x
 Exercise 6: $f(x) = \log(1 + x^2)$ for all x
 Exercise 7: $f(x) = (\log x)^2$ for $x > 0$
 Exercise 8: $f(x) = x(\log x)^2$ for $x > 0$
 Exercise 9: $f(x) = x(\log(x^2))^2 - 3x\log(x^2)$ for $x \neq 0$
 Exercise 10: $f(x) = x^{2/3}(6 - x)^{1/3}$ for $-1 \leq x \leq 7$
 Exercise 11: $f(x) = \frac{x \log x}{1+x^2}$ for $x > 0$
 Exercise 12: Theoretical

Document: 4.3 Applied Maxima and Minima

Movie:  4.3 Applied Maxima and Minima

4.3.1 Elementary Exercises on Applied Maxima and Minima

- Exercise 1: The Chicken Coop Problem
 Exercise 2: The Box Problem
 Exercise 3: The Cylindrical Can Problem
 Exercise 4: The Rectangle in a Semicircle Problem
 Exercise 5: The Isosceles Triangle in a Parabola Problem
 Exercise 6: The Isosceles Triangle in a Circle Problem
 Exercise 7: The Cone in a Hemisphere Problem
 Exercise 8: The Triangle and Semicircle Problem
 Exercise 9: The Road and Field Problem (Special Case)
 Exercise 10: The Dimmer Switch Problem
 Exercise 11: An Electric Circuit Problem

4.3.2 The General Road and Field Problem (and Deriving Snell's Law)

- The Narrow Road Version of the Road and Field Problem
 The Wide Road Version of the Road and Field Problem
 The Road and Field Problem and the Laws of Refraction
 Comparing the Wide Road Problem with the Narrow Road Problem

4.3.3 Making a Quadrilateral of Maximum Area

- Maximizing the Area of a Quadrilateral with Given Sides
 The Three Sticks Problem

4.3.4 The Ice Cream Problem: Maximum Minimum Problems About Cones

- Background Information About Cones
 Maximizing the Volume of a Cone with a Given Slant Height
 Minimizing the Slant Height of a Cone with a Given Volume
 Maximizing the Volume of a Cone with Given Surface Area
 Filling the Cone with Ice Cream

4.3.5 Introducing The Soapbox Car Problem (See Section 8.4 for the full discussion.)

Document: 4.4 Antiderivatives (Indefinite Integrals)

Movie:  4.4 Antiderivatives (Indefinite Integrals)

4.4.1 Antiderivative of a Function

4.4.2 Some Examples of Antiderivatives

- Example 1: Antiderivative with respect x of $6x$
 Example 2: Another antiderivative with respect x of $6x$
 Example 3: Antiderivative with respect x of $\cos x$
 Example 4: Antiderivative with respect x of $\frac{1}{x}$ when $x > 0$
 Example 5: Antiderivative with respect x of $\frac{1}{x}$ when $x < 0$
 Example 6: Antiderivative with respect x of $\frac{1}{x}$ when $x \neq 0$
 Example 7: Antiderivative with respect x of x^p when $p \neq -1$

4.4.3 The Key Fact About Antiderivatives

- Statement of the Key Fact
 Finding all Possible Antiderivatives of a Given Function

4.4.4 Some Examples of General Antiderivatives

Example 1: $\int x dx = \frac{x^2}{2} + c$

Example 2: $\int x^p dx = \frac{x^{p+1}}{p+1} + c$

Example 3: $\int \frac{1}{x} dx = \log|x| + c$

Example 4: $\int \cos x dx = \sin x + c$

Example 5: $\int \sin x dx = -\cos x + c$

Example 6: $\int \sec^2 x dx = \tan x + c$

Example 7: $\int \sec x \tan x dx = \sec x + c$

Example 8: $\int \tan x dx = \log|\sec x| + c$

Example 9: $\int \cot x dx = \log|\sin x| + c$

Example 10: $\int \sec x dx = \log|\sec x + \tan x| + c$

Example 11: $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

Example 12: $\int \frac{1}{1+x^2} dx = \arctan x + c$

Example 13: $\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + c$

4.4.5 Changing Variable to Find an Antiderivative

Motivating the Change of Variable Method: Example 1
Motivating the Change of Variable Method: Example 2
Motivating the Change of Variable Method: Example 3
Motivating the Change of Variable Method: Example 4
Motivating the Change of Variable Method: Example 5
Motivating the Change of Variable Method: Example 6
Motivating the Change of Variable Method: Example 7
Introducing the Change of Variable Method
Applying The Change of Variable Method

4.4.6 Some Exercises on Changing Variable

Exercise 1: $\int \sqrt{1+x^2} 2x dx$

Exercise 2: $\int \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$

Exercise 3: $\int \frac{x}{1+x^2} dx$

Exercise 4: $\int \cos^4 x \sin x dx$

Exercise 5: $\int \sqrt{\tan x} \sec^2 x dx$

Exercise 6: $\int \frac{\cos(\log x)}{x} dx$

Exercise 7: $\int e^x \sin(3e^x) dx$

Exercise 8: $\int (\log \sin x)^2 \cot x dx$

Exercise 9: $\int x\sqrt{x+3} dx$

Exercise 10: $\int \sqrt{\sin x} \cos x dx$

Exercise 11: $\int \sqrt{\sin x} \cos^3 x dx$

Exercise 12: $\int \sqrt{\sin x} \cos^5 x dx$

Exercise 13: $\int \sqrt{\cos x} \sin^5 x dx$

Exercise 14: $\int \sec^6 x \sqrt{\tan x} dx$

Exercise 15: $\int \sec^3 x \tan^5 x dx$

Exercise 16: $\int (1+x)dx$ (two ways)

Exercise 17: $\int \sin 2\theta d\theta$ (two ways)

Exercise 18: $\int \frac{1}{1-x^2} dx$

Exercise 19: $\int \sec x dx$

Exercise 20: $\int \csc x dx$

4.4.7 Antiderivatives that Involve Hyperbolic Functions

Exercise 1: $\int \cosh x dx = \sinh x + c$

Exercise 2: $\int \sinh x dx = \cosh x + c$

Exercise 3: $\int \operatorname{sech} x dx = 2 \arctan(e^x) + c$

Exercise 4: $\int \tanh x dx = \log \cosh x + c$

Exercise 5: $\int \sqrt[3]{\tanh x} \operatorname{sech}^2 x dx$

Exercise 6: $\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arcsinh} x + c$

Exercise 7: $\int \frac{\operatorname{arcsinh} x}{\sqrt{x^2+1}} dx$

Exercise 8: $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx$

Exercise 9: $\int \frac{\sqrt{\operatorname{arccosh} x}}{\sqrt{x^2-1}} dx$

Exercise 10: $\int \frac{1}{1-x^2} dx = \operatorname{arctanh} x + c$

Exercise 11: $\int \frac{1}{x\sqrt{1-x^2}} dx = -\operatorname{arcsech} x + c$

Document: 4.5 Rates of Change

Movie:  4.5 Rates of Change

4.5.1 Interpreting the Derivative as a Rate of Change

4.5.2 Some Exercises on Derivatives as Rates of Change

- Exercise 1. Inflating a Balloon: Part 1
- Exercise 2. Inflating a Balloon: Part 2
- Exercise 3. A Leaking Cone: Part 1
- Exercise 4. A Leaking Cone: Part 2
- Exercise 5. Water Evaporating from a Cone
- Exercise 6. Growth of a Bacterial Colony
- Exercise 7. Growth of Money in a Bank Account
- Exercise 8. Radioactive Decay

Document: 4.6 Motion of a Particle in a Straight Line

Movie:  4.6 Motion of a Particle in a Straight Line

4.6.1 The Position Function of a Moving Particle

4.6.2 Examples to Illustrate Position Functions

Example 1: $f(t) = t^2$ for $-1 \leq t \leq 1$

Example 2: $f(t) = t^4$ for $-1 \leq t \leq 1$

Example 3: $f(t) = t^2$ for $t \geq 0$

Example 4: $f(t) = \sin t$ for $t \geq 0$

4.6.3 Velocity, Speed, and Acceleration of a Particle

4.6.4 Some Exercises on Velocity, Speed, and Acceleration

Exercise 1: $f(t) = t^2$ at each time t

Exercise 2: $f(t) = \sin t$ at each time t in the interval $[0, 6\pi]$

Exercise 3: $f'(t) = 5t$ for each time t

Exercise 4: $f''(t) = 20$ for every t

4.6.5 Expressing Velocity and Acceleration in Terms of Position

An Example to Illustrate Velocity and Acceleration at a Point x

A Formula for Velocity in Terms of Position

Returning to the Example

A Formula for Acceleration in Terms of Position

Returning, Once Again, to the Example

4.6.6 Newton's Law

Introducing the Concept of Mass

Introducing the Concept of Force

Introduction to Newton's Law

The Role of Force when Mass is Changing: The Sticky Ball Example

The Role of Force when Velocity is Changing

Newton's Law of Motion when the Force Acts in the Direction of the Number Line

Newton's Law of Motion when the Force Acts Against the Direction of the Number Line

Units to Be Used in Newton's Law

The Kilogram, the Newton, and the Meter

The Gram, the Dyne, and the centimeter

The Pound Mass, the Poundal, and the Foot

The Slug, the Pound Force, and the Foot (Included Reluctantly)

4.6.7 Some Exercises on Newton's Law

Exercise 1: A Constant Mass Propelled by a Constant Force

Exercise 2: A Constant Mass Projected Upward Near the Ground

Exercise 3: A Sticky Ball Coasting in a Dust Cloud

Exercise 4: A Sticky Ball Coasting in a Resisting Dust Cloud

Exercise 5: Another Sticky Ball Problem

Exercise 6: A Particle Coasting in a Resisting Medium; Resistance Proportional to the Velocity

Exercise 7: A Particle Coasting in a Resisting Medium; Resistance Proportional to the Square of the Velocity

Exercise 8: A Rocket Problem

Exercise 9: A Particle Moving Away from the Earth

Exercise 10: A Relativistic Problem

Overview of Chapter 5: The Mean Value Theorem and its Applications

Document: 5.1 The Mean Value Theorem

Movie:



5.1 The Mean Value Theorem

5.1.1 Introduction to the Mean Value Theorem

Why Do We Need the Mean Value Theorem?

A Sneak Preview of the Mean Value Theorem

Statement of the Mean Value Theorem

The Speeding Ticket Problem

5.1.2 Rolle's Theorem

The Statement of Rolle's Theorem

Two Important Ingredients Needed for Rolle's theorem

A Brief Restatement of Fermat's theorem

A Brief Restatement of the Theorem on Maxima and Minima of Continuous Functions

Proof of Rolle's Theorem

A Two Function Version of Rolle's Theorem

Proof of the Mean Value Theorem

5.1.3 Proving the Positive Derivative Principle

Proof of Assertion 1

Proof of Assertion 2

Proof of Assertion 3

Proof of Assertion 4

Proof of Assertion 5

5.1.4 Some Exercises on the Mean Value Theorem

- Exercise 1: A function with a maximum
- Exercise 2: The derivative of a strictly increasing function
- Exercise 3: Reversing the endpoints of the interval
- Exercise 4: A condition for a function to be one-one
- Exercise 5: Using the inequality $|f'(x)| \leq 1$
- Exercise 6: When the inequality $|f(t) - f(x)| \leq |t - x|^2$ holds
- Exercise 7: A condition for two functions to be \sin and \cos
- Exercise 8: Derivatives have an intermediate value property
- Exercise 9: A two function version of Exercise 8

Document: 5.2 Approximating a Function with Polynomials

Movie:  5.2 Approximating a Function with Polynomials

5.2.1 Introduction to Polynomials

- Definition of a Polynomial
- Expanding $(1 + x)^8$: Motivating the Binomial Theorem
- The Binomial Theorem

5.2.2 The Coefficients of a General Polynomial

- Special Notation for Higher Derivatives of a Function
- Finding the Coefficients of a Given Polynomial
- The Degree of a Polynomial
- Recentering the Terms of a Polynomial

5.2.3 Taylor Polynomials of a Function

- Definition of The Taylor Polynomials

5.2.4 Some Examples of Taylor Polynomials

- Example 1: $f(x) = 2 - 4x + 3x^2 + 7x^3 + 5x^4$ for each x
- Example 2: $f(x) = \frac{1}{1+x^2}$ for each x
- Example 3: $f(x) = \frac{1}{1+x^2}$, Taylor polynomials centered at 1
- Example 4: Using a computer algebra system to find Taylor polynomials
- Example 5: Another application of a computer algebra system


5.2.5 Finding The Remainder Term

- Introducing the Remainder Term of a Taylor Polynomial
- A Quick Review of Rolle's Theorem
- A Version of Rolle's Theorem for the Second Derivative
- A Version of Rolle's Theorem for the Third Derivative
- A Version of Rolle's Theorem for the Fourth Derivative
- Motivating the Higher Derivative Form of the Mean Value Theorem: A Mean Value Theorem for the Fourth Derivative
- The Higher Derivative Form of the Mean Value Theorem (Sometimes Called the Taylor Mean Value Theorem)

5.2.6 Some Applications of the Taylor Mean Value Theorem

- Finding an Approximation to e
- The Number e is Irrational
- Finding an Approximation to e^3
- Finding an approximation to $\log\left(\frac{3}{2}\right)$
- Finding an approximation to $\log\left(\frac{1}{2}\right)$
- Finding An Approximation to $\cos 1$
- The Number $\cos 1$ Is Irrational

Document: 5.3 Indeterminate Forms

Movie:  5.3 Indeterminate Forms

5.3.1 Introduction to Indeterminate Forms

5.3.2 Some Examples to Illustrate Indeterminate Forms

Example 1: $\lim_{x \rightarrow 0} \frac{3x}{x} = 3$

Example 2: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Example 3: $\lim_{x \rightarrow 0^+} x(\log x) = 0$

Example 4: $\lim_{x \rightarrow \infty} \frac{(\log x)^3}{x} = 0$

Example 5: $\lim_{x \rightarrow 0} (1 + 2x)^{1/x} = e^2$

Example 6: $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x + 1} - \sqrt{x^2 - 2x + 7}) = \frac{5}{2}$

5.3.3 L'Hôpital's rule

Introducing L'Hôpital's rule

More Careful Statement of L'Hôpital's rule

Some Remarks About L'Hôpital's rule

The Rule Works for One-Sided and Two-Sided Limits

The Limit May Be Finite or Infinite

The Case in Which $\lim_{x \rightarrow a} g(x) = \infty$

A Brief History of L'Hôpital's rule

A Special Case of L'Hôpital's Rule

Example 1 Showing Use of the Special Case of L'Hôpital's Rule

Example 2 Showing Use of the Special Case of L'Hôpital's Rule

Proof of the Special Case of L'Hôpital's Rule

5.3.4 Exercises on Indeterminate Forms

Exercise 1: $\lim_{x \rightarrow \infty} \frac{3x - 7}{2x + 5}$

Exercise 2: $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

Exercise 3: $\lim_{x \rightarrow 0} \frac{e^x \sin 5x - \sin 3x}{x}$

Exercise 4: $\lim_{x \rightarrow 0} \left(\frac{\tan x - x}{x - \sin x} \right)$

Exercise 5: $\lim_{x \rightarrow 0} \left(\frac{x - \sin x}{x^3} \right)$

Exercise 6: $\lim_{x \rightarrow 0^+} x \log x$

Exercise 7: $\lim_{x \rightarrow 1} \frac{\log x}{x - 1}$

Exercise 8: $\lim_{x \rightarrow \infty} \frac{\log x}{x}$

Exercise 9: $\lim_{x \rightarrow \infty} \frac{(\log x)^2}{x}$

Exercise 10: $\lim_{x \rightarrow \infty} \frac{(\log x)^p}{x} = 0$

Exercise 11: $\lim_{x \rightarrow \infty} (\log(3x + 2) - \log(2x - 5))$

Exercise 12: $\lim_{x \rightarrow \infty} \frac{\log(x + 2)}{\log(x - 5)}$

Exercise 13: $\lim_{x \rightarrow \infty} \left(\frac{(\log(3x + 2))^2 - (\log(2x - 5))^2}{\log x} \right)$

Exercise 14: $\lim_{x \rightarrow \infty} \frac{\exp(\sqrt{\log x})}{x}$

Exercise 15: $\lim_{x \rightarrow \pi/2} (\sin x)^{\tan x}$

Exercise 16: $\lim_{x \rightarrow \infty} x^{(\log x)/x}$

Exercise 17: $\lim_{x \rightarrow 0} ((1 + px)^{1/x})$

Exercise 18: $\lim_{x \rightarrow 0} \frac{e - (1 + x)^{1/x}}{x}$

Exercise 19: $\lim_{x \rightarrow \infty} ((\log(x + 1))^2 - (\log x)^2)$

Exercise 20: $\lim_{x \rightarrow \infty} \frac{(x + 1)^{\log(x+1)}}{x^{\log x}}$

Exercise 21: $\lim_{x \rightarrow \infty} \left(\frac{\log(x + 1)}{\log x} \right)^x$

Exercise 22: $\lim_{x \rightarrow \infty} \frac{x e^{\sin x}}{\log x}$

5.3.5 An Important Limit: $\lim_{x \rightarrow \infty} x \left(1 - \frac{x^p}{(x+1)^p} \right)$

Overview of Chapter 6: Integrals

Document: 6.1 Introducing Integrals as Antiderivatives

Movie:  6.1 Introducing Integrals as Antiderivatives

6.1.1 Preliminary Note: This Movie Takes the Fast Track into Integral Calculus

6.1.2 Defining Integrals Using Antiderivatives

Reviewing a Property of Antiderivatives

Defining the Symbol $\int_a^b f(x) dx$

Notation for Taking a Function Between Limits

The Symbol x is Not Important

6.1.3 Some Examples to Illustrate the Definition of an Integral

Example 1: $\int_2^5 x dx$

Example 2: $\int_0^{\pi/2} \cos x dx$

Example 3: $\int_2^9 \frac{1}{x} dx$

Example 4: $\int_0^{\pi/4} \sec^2 x dx$

Example 5: $\int_0^{\pi/4} \sec x \tan x dx$

Example 6: $\int_0^{\pi/4} \sec x dx$

Example 7: $\int_{-1}^2 2x\sqrt{1+x^2} dx$

6.1.4 Linearity and Additivity of the Integral

Linearity of the Integral

Additivity of the Integral

The Symbol \int_b^a when $a < b$

6.1.5 Using Integrals to Find Area

The Area Under the Graph of a Nonnegative Function: Historical Approach Using Infinitesimals

The Area Under the Graph of a Nonnegative Function Without Using Infinitesimals

Area of the Region Between Two Graphs

Area Between the Graph of a Negative Function and the x -Axis

6.1.6 Some Exercises on Area

Exercise 1: The region between $y = 4 - x^2$ and the x -axis

Exercise 2: A triangular region

Exercise 3: Region between $y = x^3 - 3x^2 + 2$ and $y = -x^2 + 3x + 2$

Exercise 4: Region between $y = \sin x$ and $y = \cos x$

Exercise 5: Region between $y = \sin x$ and $y = \sin 2x$

Exercise 6: Region between $y = \sin x$ and $y = \sqrt{\sin x} \cos x$

6.1.7 Derivatives of Integrals: The Equation $\frac{d}{dx} \int_a^x f(t) dt = f(x)$

6.1.8 Exercises on Derivatives of Integrals

Exercise 1: $\frac{d}{dx} \int_1^x \sqrt{1+t+t^4} dt$

Exercise 2: $\frac{d}{dx} \int_1^x \sqrt{1+t+t^4} dt$

Exercise 3: $\frac{d}{dx} \int_2^{\sin x} \sqrt{1+t+t^4} dt$

Exercise 4: $\frac{d}{dx} \int_2^{\log x} \sqrt[3]{1 + \sin^2 t} dt$

Exercise 5: $\frac{d}{dx} \int_{\exp(\sin x)}^5 \sqrt{1 + t^2} dt$

Exercise 6: $\frac{d}{dx} \int_{\sin x}^{\exp(x^2)} \sqrt{1 + t^4} dt$

Document: 6.2 Riemann Sums

Movie:  **6.2 Riemann Sums**

6.2.1 Summation Notation

Introducing Summation Notation
Some Simple Examples to Illustrate Summation Notation

Example 1: $\sum_{j=3}^5 j^3$

Example 2: $\sum_{j=0}^7 (-1)^j$

Example 3: $\sum_{j=1}^n 4$

Arithmetical Rules for Summation

Working Out the Sum $\sum_{j=1}^n j$

Another Way of Working Out $\sum_{j=1}^n j$

Working Out the Sum $\sum_{j=1}^n j^2$

Working Out the Sum $\sum_{j=1}^n j^3$

Using a Computer Algebra System to Work Out $\sum_{j=1}^n j^p$

6.2.2 Introduction to Riemann Sums

Motivating Riemann Sums
Definition of a Partition
Definition of a Riemann Sum
Regular Partitions
Darboux's Theorem
Left Sums, Right Sums, and Midpoint Sums
Left Sums
Right Sums
Midpoint Sums

6.2.3 Some Examples to Illustrate Darboux's Theorem

Example 1: $\int_0^1 x dx$


Example 2: $\int_0^1 x^2 dx$

Example 3: $\int_a^b x^2 dx$

Example 4: $\int_0^1 \sqrt{x} dx$

Example 5: $\int_0^1 \sqrt[3]{x^2} dx$

Document: 6.3 Riemann Sums with a Computer Algebra System

Movie:  **6.3 Riemann Sums with a Computer Algebra System**

6.3.1 Introductory Comment

6.3.2 Setting up The Riemann Sums

Supplying the Regular Partition to a Computer Algebra System
Introducing a Temporary Function f
Defining the Left Sum of a Function
Defining the Right Sum of a Function
Defining the Trapezoidal Sum of a Function
Defining Midpoint Sum of a Function
Defining the Simpson Sum of a Function
Motivation of the Simpson Sum

6.3.3 Numerical Approximations to Integrals

Summary of the Definitions
Using the Sums to Estimate $\int_0^1 \sqrt[3]{1+x^2} dx$
Using the Sums to Estimate $\int_0^1 \sqrt[3]{1-x^2} dx$
Obtaining Arrays of Approximating Sums Automatically

Document: 6.4 Using Riemann Sums to Define an Integral

Movie:  6.4 Using Riemann Sums to Define an Integral

6.4.1 Our Objective in this Section

6.4.2 A Quick Review of Riemann Sums

Bounded Functions
Definition of a Partition
Definition of a Riemann Sum
Regular Partitions

6.4.3 Squeezing a Function, Integrability, and the Integral

Motivating the Idea of Squeezing
Definition of a Squeezing Pair of Sequences
A Key Fact About a Squeezing Pair of Sequences
Integrability and the Integral

6.4.4 Some Examples to Illustrate Integrability

Example 1: The Integral $\int_0^1 x dx$
Example 2: The Integral $\int_0^1 x^2 dx$
Example 3: Increasing Functions Are Integrable
Example 4: Decreasing Functions Are Integrable
Example 5: Continuous Functions Are Integrable
Example 6: A Function that Fails to be Integrable

6.4.5 Some Facts About the Integral

Linearity of the Integral
Nonnegativity of the integral
Additivity of the Integral
Darboux's Theorem

6.4.6 The Fundamental Theorem of Calculus

Part 1 of the Fundamental Theorem of Calculus
Part 2 of the Fundamental Theorem of Calculus

6.4.7 Optional Item: Error Estimates for the Simpson Sum (Not included in the video)

Background for the Error Estimates
An Example to Illustrate the Error Estimates

Overview of Chapter 7: Evaluating Integrals

Document: 7.1 Evaluating Integrals by Substitution

Movie:  7.1 Evaluating Integrals by Substitution

7.1.1 Some Common Antiderivatives

The Antiderivative $\int x^p dx$ when $p \neq -1$

The Antiderivative $\int e^x dx$

The Antiderivative $\int x^p dx$ when $p = -1$

The Antiderivative $\int \cos x dx$

The Antiderivative $\int \sin x dx$

The Antiderivative $\int \sec^2 x dx$

The Antiderivative $\int \sec x \tan x dx$

The Antiderivative $\int \tan x dx$

The Antiderivative $\int \cot x dx$

The Antiderivative $\int \sec x dx$

The Antiderivative $\int \frac{1}{\sqrt{1-x^2}} dx$

The Antiderivative $\int \frac{1}{1+x^2} dx$

The Antiderivative $\int \frac{1}{x\sqrt{x^2-1}} dx$

The List of Antiderivatives

7.1.2 Changing Variable to Calculate an Integral

Introducing the Change of Variable Method

Applying the Change of Variable Method

7.1.3 Some Exercises on the Change of Variable Method

Exercise 1: $\int_0^{\pi/2} \sin^2 x \cos x dx$

Exercise 2: $\int_0^1 \sqrt{1+x^2} 2x dx$

Exercise 3: $\int_0^1 \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$

Exercise 4: $\int_1^2 \frac{x}{1+x^2} dx$

Exercise 5: $\int_0^{\pi} \cos^4 x \sin x dx$

Exercise 6: $\int_0^{\pi/4} \sqrt{\tan x} \sec^2 x dx$

Exercise 7: $\int_0^{\pi/3} \tan^2 x dx$

Exercise 8: $\int_0^{\pi/3} \tan^3 x dx$

Exercise 9: $\int_0^{\pi/4} \tan^4 x dx$

Exercise 10: $\int_1^{\exp(\pi/3)} \frac{\cos(\log x)}{x} dx$

Exercise 11: $\int_{\log(\pi/12)}^{\log(\pi/6)} e^x \sin(3e^x) dx$

Exercise 12: $\int_0^1 x\sqrt{x+3} dx$

Exercise 13: $\int_0^{\pi/2} \sqrt{\sin x} \cos x dx$

Exercise 14: $\int_0^{\pi/2} \sqrt{\sin x} \cos^3 x dx$

Exercise 15: $\int_0^{\pi/2} \sqrt{\sin x} \cos^5 x dx$

Exercise 16: $\int_0^{\pi/2} \sqrt{\cos x} \sin^5 x dx$

Exercise 17: $\int_0^{\pi/2} \cos^2 x dx$

Exercise 18: $\int_0^{\pi/2} \sin^4 x \cos^2 x dx$

Exercise 19: $\int_0^{\pi} \sqrt[3]{1+2\sin^2 x + \sin^4 x} \cos x dx$

Exercise 20: $\int_0^{\pi/4} \sec^6 x \sqrt{\tan x} dx$

Exercise 21: $\int_0^{\pi/3} \sec^3 x \tan^5 x dx$

Exercise 22: $\int_0^{\pi/3} \sec x dx$

Exercise 23: $\int_0^{1/2} \frac{\sqrt[3]{\arcsin x}}{\sqrt{1-x^2}} dx$

Exercise 24: $\int_1^{\sqrt{3}} \frac{1}{(1+x^2) \arctan x} dx$

Exercise 25: $\int_0^1 \frac{\arctan x}{(1+x^2) \sqrt{1+(\arctan x)^2}} dx$

Exercise 26: $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1} \operatorname{arcsec} x} dx$

Document: 7.2 Evaluating Integrals by Parts

Movie:  7.2 Evaluating Integrals by Parts

7.2.1 Introduction to Integration by Parts

7.2.2 Some Examples to Illustrate Integration by Parts

Example 1: $\int_0^{\pi/2} x \cos x dx$

Example 2: $\int_0^1 x e^{3x} dx$

Example 3: $\int_0^{\pi/2} \cos^2 x dx$

7.2.3 Explaining Integration by Parts

Explaining Integration by Parts for Integrals
Explaining Integration by Parts for Antiderivatives

7.2.4 Exercises on Integration by Parts

Exercise 1

Exercise 1 Part a: $\int_0^{\pi/2} x^2 \cos x dx$

Exercise 1 Part b: $\int x^2 \cos x dx$

Exercise 2

Exercise 2 Part a: $\int_1^2 x \log x dx$

Exercise 2 Part b: $\int x \log x dx$

Exercise 3

Exercise 3 Part a: $\int_1^2 x(\log x)^2 dx$

Exercise 3 Part b: $\int x(\log x)^2 dx$

Exercise 4

Exercise 4 Part a: $\int_1^2 x(\log x)^3 dx$

Exercise 4 Part b: $\int x(\log x)^3 dx$

Exercise 5: $\int_1^2 \log x dx$

Exercise 6: $\int_0^{\pi^2/4} \cos \sqrt{x} dx$

Exercise 7: $\int_0^1 \arctan x dx$

Exercise 8: $\int_0^1 x \arctan x dx$

Exercise 9: $\int_0^{1/2} \arcsin x dx$

Exercise 10: $\int_0^{\pi/2} x \sin x \cos x dx$

Exercise 11: $\int_0^1 x \arcsin x dx$

Exercise 12: $\int_0^\pi e^x \cos x dx$

Exercise 13: $\int_0^{\pi/3} \sec^3 x dx$

Exercise 14: $\int_0^{\log \sqrt{3}} \operatorname{sech}^3 x dx$

Exercise 15: $\int_0^{2\pi} \cos mx \cos nx dx$

7.2.5 Reduction Formulas

Introduction to Reduction Formulas

Example 1: A Reduction Formula for the Integral $\int_1^2 x(\log x)^n dx$

Example 2: A Reduction Formula for the Antiderivative $\int x(\log x)^n dx$

Example 3: A Reduction Formula for the Integral $\int_0^{\pi/2} x^n \cos x dx$

Example 4: A Reduction Formula for the Antiderivative $\int x^n e^x dx$

Example 5: A Reduction Formula for the Antiderivative $\int \cos^n x dx$

Example 6: A Reduction Formula for the Integral $\int_0^{\pi/2} \cos^n x dx$

Example 7: A Reduction Formula for the Antiderivative $\int \sin^n x dx$

Example 8: A Reduction Formula for the Integral $\int_0^{\pi/2} \sin^n x dx$

Example 9: A Reduction Formula for the Antiderivative $\int \tan^n x dx$

Example 10: A Reduction Formula for the Antiderivative $\int \cot^n x dx$

Example 11: A Reduction Formula for the Antiderivative $\int \sec^n x dx$

Example 12: A Reduction Formula for the Integral $\int_0^{\pi/4} \sec^n x dx$

7.2.6 Wallis' Formula: $\lim_{n \rightarrow \infty} \frac{2^{2n} (n!)^2}{\sqrt{n} (2n)!} = \sqrt{\pi}$

Introduction to Wallis' Formula

A Return to the Integral $\int_0^{\pi/2} \cos^n x dx$

Deriving Wallis' Formula

Document: 7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

Movie Option 1:  7.3 Evaluating Integrals Using Trigonometric Substitutions Only

Movie Option 2:

 7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

7.3.1 Preliminary Notes

Introduction to this Section

How Do I Know Whether to Use Trig or Hyperbolic Substitutions?

How Do I Know Whether a Given Integral Is of Type 1, 2, or 3?

7.3.2 Substitutions Involving sin or tanh

Introduction to the sin Substitution

An Example to Illustrate the sin Substitution

Introduction to the tanh Substitution

An Example to Illustrate the tanh Substitution

Integrals of Expressions Involving $\sqrt{a^2 - x^2}$

7.3.3 Substitutions Involving sec or cosh

Introduction to the sec Substitution

An Example to Illustrate the sec Substitution

Introduction to the cosh Substitution

An Example to Illustrate the cosh Substitution

Integrals of Expressions Involving $\sqrt{x^2 - a^2}$

7.3.4 Substitutions Involving tan or sinh

Introduction to the tan Substitution

An Example to Illustrate the tan Substitution

Introduction to the \sinh Substitution
An Example to Illustrate the \sinh Substitution
Integrals of Expressions Involving $\sqrt{a^2 + x^2}$

7.3.5 Exercises on Trigonometric and Hyperbolic Substitutions

Exercise 1: $\int_0^{3/2} \sqrt{9 - x^2} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 2: $\int_0^3 \sqrt{9 - x^2} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution: **Omitted**

Exercise 3: $\int_0^{5/2} \frac{1}{\sqrt{25 - x^2}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 4: $\int_0^3 \frac{1}{9 + x^2} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 5: $\int_{\sqrt{x}}^2 \frac{x^2}{\sqrt{x^2 - 1}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 6: $\int_{3\sqrt{2}}^6 \frac{1}{(x^2 - 9)^{3/2}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 7: $\int_0^1 \frac{x}{(1 + x^2)^{3/2}} dx$

Exercise 8: $\int_0^{1/2} \frac{x^2}{\sqrt{1 - x^2}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 9: $\int_0^{1/2} \frac{x}{\sqrt{1 - x^2}} dx$

Exercise 10: $\int_{1/2}^{1/\sqrt{2}} \frac{1}{x\sqrt{1 - x^2}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 11: $\int_0^1 \frac{x^2}{(1 + x^2)^{3/2}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 12: $\int_{3\sqrt{2}}^6 \frac{1}{x\sqrt{x^2 - 9}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 13: $\int_{3\sqrt{2}}^6 \frac{1}{x^2\sqrt{x^2 - 9}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 14: $\int_{3\sqrt{2}}^6 \frac{1}{x^4\sqrt{x^2 - 9}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 15: $\int_{3\sqrt{2}}^6 \frac{1}{\sqrt{x^2 - 9}} dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 16: $\int_{3\sqrt{2}}^6 \frac{x}{\sqrt{x^2 - 9}} dx$

Exercise 17: $\int_{3\sqrt{2}}^6 \frac{x^3}{\sqrt{x^2 - 9}} dx$

Exercise 18: $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx$

Evaluation Using a Trigonometric Substitution
 Evaluation Using a Hyperbolic Substitution

Exercise 19: $\int_1^2 \frac{\sqrt{x^2 - 1}}{x^4} dx$

Evaluation Using a Trigonometric Substitution
 Evaluation Using a Hyperbolic Substitution

Exercise 20: $\int_5^{3+2\sqrt{3}} \frac{1}{\sqrt{x^2 - 6x + 13}} dx$

Evaluation Using a Trigonometric Substitution
 Evaluation Using a Hyperbolic Substitution

Exercise 21: $\int_{1/2}^2 \frac{1}{(2x^2 - 2x + 5)^{3/2}} dx$

Evaluation Using a Trigonometric Substitution
 Evaluation Using a Hyperbolic Substitution

Exercise 22: $\int_{1+3\sqrt{2}}^7 \frac{1}{\sqrt{x^2 - 2x - 8}} dx$

Evaluation Using a Trigonometric Substitution
 Evaluation Using a Hyperbolic Substitution

Exercise 23: $\int_3^5 \sqrt{6x - 5 - x^2} dx$

Evaluation Using a Trigonometric Substitution
 Evaluation Using a Hyperbolic Substitution: **Omitted**

Exercise 24: $\int_1^{4/3} \frac{1}{(18x - 9x^2 - 5)^{3/2}} dx$

Evaluation Using a Trigonometric Substitution
 Evaluation Using a Hyperbolic Substitution

Document: 7.4 Integration of Rational Functions

Movie:  7.4 Integration of Rational Functions

7.4.1 Background on Rational Functions

Introducing Rational Functions
 Partial Fraction Expansions of Rational Functions

7.4.2 Some Exercises on Integration of Rational Functions

Exercise 1: $\int \frac{x + 23}{x^2 - 3x - 10} dx$

Exercise 2: $\int_0^1 \frac{3x^2 + 8x + 7}{(x + 1)(x + 2)^2} dx$

Exercise 3: $\int_{-1}^1 \frac{x^2 + x + 2}{(x + 3)(x^2 + 2x + 5)} dx$

Exercise 4: $\int_{-1}^1 \frac{2x - 2}{(x + 3)(x^2 + 2x + 5)} dx$

Exercise 5: $\int_{-1}^1 \frac{x^2 + 5x - 2}{(x + 3)(x^2 + 2x + 5)} dx$

Exercise 6: $\int_0^{\pi/4} \sqrt{\tan x} dx$

Exercise 7: $\int_0^{\pi/4} \sqrt[3]{\tan x} dx$

7.4.3 Integrating Rational Functions of \cos and \sin

7.4.4 Exercises on Rational Functions of $\cos \theta$ and $\sin \theta$

Exercise 1: $\int_0^{\pi/2} \frac{1}{\sin \theta + \cos \theta} d\theta$

Exercise 2: An Alternative Approach to $\int_0^{\pi/2} \frac{1}{\cos \theta + \sin \theta} d\theta$

Exercise 3: $\int_0^{\pi/2} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta$

Exercise 4: $\int_0^{\pi/2} \frac{\sin \theta}{1 + \cos \theta + \sin \theta} d\theta$

Document: 7.5 Evaluating Improper Integrals

Movie:  7.5 Evaluating Improper Integrals

7.5.1 Introduction to Improper Integrals

Example 1: Motivating The integral $\int_0^1 \frac{1}{\sqrt{x}} dx$

Example 2: Motivating The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

Example 3: Motivating The integral $\int_1^\infty \frac{1}{x^2} dx$

Definition of an Improper Integral

Definition of an Integral that Improper at its Right Endpoint

Definition of an Integral that Improper at its Left Endpoint

Convergence and Divergence of an Improper Integral

7.5.2 Some Examples of Improper Integrals

Example 1: $\int_0^1 \frac{1}{\sqrt{x}} dx$

Example 2: $\int_1^\infty \frac{1}{\sqrt{x}} dx$

Example 3: $\int_1^\infty \frac{1}{x^2} dx$

Example 4: $\int_0^\infty \cos x dx$

Example 5: $\int_0^1 \arcsin x dx$

7.5.3 Some Exercises on Improper Integrals

Exercise 1: $\int_0^\infty \frac{1}{(1+x^2)^{3/2}} dx$

The More Careful Approach

The Quick Approach

Exercise 2: $\int_2^\infty \frac{1}{x\sqrt{x^2-1}} dx$

The More Careful Approach

The Quick Approach

Exercise 3: $\int_1^2 \frac{1}{x\sqrt{x^2-1}} dx$

The More Careful Approach

The Quick Approach

Exercise 4: $\int_0^{\pi/2} \tan x dx$

The More Careful Approach

The Quick Approach

Exercise 5: $\int_0^{\pi/2} \sqrt{\tan x \sin x} dx$

The More Careful Approach

The Quick Approach

Exercise 6: $\int_0^{\pi/2} \frac{x \cos x - \sin x}{x^2} dx$

The More Careful Approach

The Quick Approach: **Omitted**

Exercise 7: $\int_0^\infty e^{-x} \sin x dx$

The More Careful Approach

The Quick Approach

Exercise 8: $\int_0^1 \frac{1}{x^p} dx$

Exercise 9: $\int_1^\infty \frac{1}{x^p} dx$

Exercise 10: $\int_2^\infty \frac{1}{x(\log x)^p} dx$

Exercise 11: $\int_0^2 \frac{1}{(x-1)^{1/3}} dx$

Exercise 12: $\int_0^1 \log x dx$

Document: 7.6 Convergence of Improper Integrals

Movie:  7.6 Convergence of Improper Integrals

7.6.1 Introduction to This Section

7.6.2 Convergence of Integrals of Nonnegative Functions

- An Fundamental Principle About Integrals of Nonnegative Functions
- Warning
- An Example to Illustrate The Fundamental Principle
- Introducing The Comparison Test for Improper Integrals
- The Comparison Test for Improper Integrals
- An Example to Illustrate the Comparison Test
- A Second Example to Illustrate the Comparison Test
- Introduction to the Limit Version of the Comparison Test
- Statement of the Limit Comparison Test
- Another Way of Looking at the Limit Comparison Test

7.6.3 Exercises on the Comparison Test

Exercise 1: $\int_1^{\infty} \frac{x}{x^3 - 3x^2 + 3x + 7} dx$

Exercise 2: $\int_0^1 \frac{1}{\sqrt{x} \cos x} dx$

Exercise 3: $\int_1^{\infty} \frac{\log x}{x^2} dx$

Exercise 4: $\int_1^{\infty} \frac{1}{\sqrt[3]{x^2 + 5x + 2}} dx$

Exercise 5: $\int_0^1 \frac{\sin^2 x}{x^{5/2}} dx$

Exercise 6: $\int_1^{\infty} \frac{\sqrt{x}}{x^2 - x + 1} dx$

Exercise 7: $\int_0^{\pi/2} \sqrt{\tan x} dx$

Exercise 8: $\int_1^2 \frac{1}{\log x} dx$

Exercise 9: $\int_0^{\pi/2} \log(\sin x) dx$

Exercise 10: $\int_1^{\infty} x^{a-1} e^{-x} dx$

Exercise 11: $\int_0^1 x^{a-1} e^{-x} dx$

Note On the Final Three Exercises of this Group

Exercise 12: $\int_2^{\infty} \frac{1}{(\log x)^{\log x}} dx$

Exercise 13: $\int_3^{\infty} \frac{1}{(\log \log x)^{\log x}} dx$

Exercise 14: $\int_3^{\infty} \frac{1}{(\log x)^{\log \log x}} dx$

7.6.4 Improper Integrals of Functions that Can Change Sign

- Absolute Convergence of an Improper Integral
- Every Absolutely Convergent Integral Must Converge
- Conditional Convergence of an Improper Integral

7.6.5 Exercises on Absolute and Conditional Convergence of Improper Integrals

Exercise 1: $\int_1^{\infty} \frac{\sin x}{x^2} dx$ and $\int_1^{\infty} \frac{\cos x}{x^2} dx$

Exercise 2: $\int_1^{\infty} \frac{\sin x}{x} dx$

Exercise 3: $\int_1^{\infty} \frac{\sin cx}{x^p} dx$

Exercise 4: $\int_1^{\infty} \frac{\cos cx}{x^p} dx$

Exercise 5: $\int_1^{\infty} \frac{\sin^2 x}{x} dx$

Exercise 6: $\int_1^{\infty} \frac{|\sin x|}{x} dx$

Exercise 7: Conditional Convergence of $\int_1^{\infty} \frac{\sin x}{x} dx$

Overview of Chapter 8: Some Applications of Derivatives and Integrals

Document: 8.1 Using Integrals to Find Volume

Movie:  8.1 Using Integrals to Find Volume

8.1.1 Volume by the Method of Slicing

8.1.2 Exercises on the Method of Slicing

- Exercise 1: Volume of a Cone
- Exercise 2: Volume of a Pyramid
- Exercise 3: Volume of a Ball
- Exercise 4: A Variation on the Cone Problem
- Exercise 5: Rotating a Plane Region Around the x -Axis
- Exercise 6: A Specific Region Rotated Around the x -Axis
- Exercise 7: A Return to the Volume of a Ball Exercise
- Exercise 8: Volume of a Bagel
- Exercise 9: An Apple Without its Core
- Exercise 10: Rotating a region bounded by $y = \sin x$ and $y = \cos x$ about the x -axis

8.1.3 Volume by the Method of Shells

- Introducing the Shell Method
- Finding the Volume of a Cylindrical Shell
- Returning to Our Introduction

8.1.4 Exercises on the Method of Shells

- Exercise 1
 - Using the Slicing Method to Find this Volume
 - Using the Shell Method to Find this Volume
- Exercise 2
 - Using the Slicing Method to Find this Volume
 - Using the Shell Method to Find this Volume
- Exercise 3
- Exercise 4
 - Using the Slicing Method to Find this Volume
 - Using the Shell Method to Find this Volume
- Exercise 5
 - Using the Slicing Method to Find this Volume
 - Using the Shell Method to Find this Volume
- Exercise 6: Using the Shell Method to Find the Volume of a Bagel

Document: 8.2 Work Done by a Force

Movie:  8.2 Work Done by a Force

8.2.1 Work Done by a Constant Force

- Introducing the Units of Work
- Lifting a Mass Near the Surface of the Earth

8.2.2 Work Done by a Variable Force

- Introducing the Formula for Work Done by a Variable Force

8.2.3 Exercises on Work Done by a Force

- Exercise 1: Stretching a Piece of Elastic
- Exercise 2: Lifting a Leaking Bag of Flour

Exercise 3: A Crane Lifting a Leaky Bag of Sand
Exercise 4: Lifting a Constant Mass from the Ground to a Specified Distance from the Earth

8.2.4 Work Done by a Force Acting on a Moving Particle

Review of the Discussion of Velocity and Acceleration in Terms of Position
Work Done by a Force Acting on a Moving Particle

8.2.5 Exercises on Work Done by a Force Acting on a Particle

Exercise 1: Kinetic Energy of a Particle with Constant Mass
Exercise 2: Projecting a Particle from the Earth
Exercise 3: A Relativistic Formula for Kinetic Energy
Einstein's Mass-Energy Relationship

Document: 8.3 Parametric and Polar Curves

Movie:



8.3 Parametric and Polar Curves

8.3.1 Parametric Curves

Motivating the Idea of a Parametric Curve
Definition of a 2D Parametric Curve

8.3.2 Some Examples of Parametric Curves

Example 1: A Curve that Runs in a Parabola
Example 2: A Restricted Form of the Curve in Example 1
Example 3: Moving Through the Parabola Several Times
Example 4: A Curve with a Loop
Example 5: A Fish Curve
Example 6: A Particle Travelling Counter Clockwise in a Circle
Example 7: A Particle Travelling Clockwise in a Circle
Example 8: A Spiral Curve
Example 9: An Exponential Spiral Curve
Example 10: The Cycloid

8.3.3 Distance Travelled along a Curve

8.3.4 Exercises on Curve Length

Exercise 1: Length of a Circle
Exercise 2: Going Twice Around a Circle
Exercise 3: Length of a Spiral Curve
Exercise 4: Length of an Exponential Spiral Curve
Exercise 5: Length of a Cycloid
Exercise 6: Length of an Ellipse

8.3.5 Area of a Surface of Revolution

8.3.6 Exercises on Surface of Revolution

Exercise 1: Area of a Sphere
Exercise 2: Area of a Cone
Exercise 3: Area of a Paraboloid
Exercise 4: Rotating the Graph of \sin
Exercise 5: Area of a Circular Ellipsoid

8.3.7 Polar Coordinates

Introduction to Polar Coordinates
Polar Coordinates are not Unique
A Relationship Between Polar Coordinates and Rectangular Coordinates
Existence of Polar Coordinates of Any Given Point
Polar Graphs

8.3.8 Exercises on Polar Coordinates

Exercise 1: Finding a Point with Given Polar Coordinates
Exercise 2: Finding Polar Coordinates of a Given Point
Exercise 3: Polar Equation of a Circle
Exercise 4: Polar Equation of a Vertical Line
Exercise 5: Polar Equation of a Horizontal Line
Exercise 6: Polar Equation of a Line Through the Origin
Exercise 7: Polar Equation of a Parabola
Exercise 8: Polar Equation of a Circle with Center at $(1,0)$
Exercise 9: Polar Equation of a Spiral Graph

Exercise 10: The Polar Graph $r = \frac{1}{\theta}$

Exercise 11: The Polar Graph $r = \frac{1}{\sqrt{\theta}}$

Exercise 12: The Polar Graph $r = \cos 2\theta$

Exercise 13: The Polar Graph $r = \sin 3\theta$

Exercise 14: The Polar Graph $r = \cos 3\theta$

Exercise 15: The Polar Graph $r = 1 + \cos \theta$

Exercise 16: The Polar Graph $r = 1 + 2 \cos \theta$

Exercise 17: A Computer Generated Polar Graph

8.3.9 Length of a Polar Graph

Introducing the Formula for Length of a Polar Graph

Example 1: Length of a Petal of the Graph $r = \cos 3\theta$.

Example 2: Length of a Cardioid

Example 3: Length of a Limacon

8.3.10 Area Bounded by a Polar Graph

8.3.11 Exercises on Area Bounded by a Polar Graph

Exercise 1: Area of a Petal of the Graph $r = \cos 3\theta$

Exercise 2: Area Enclosed by Cardioid

Exercise 3: Area Enclosed by a Spiral

Exercise 5: Area Enclosed by an Inward Spiral

Document: 8.4 The Soapbox Problem

Movie:  8.4 The Soapbox Problem

8.4.1 Introducing The Soapbox Car Problem

8.4.2 Preliminary Discussion: Maximizing a Special Kind of Rational Function

8.4.3 Finding the Kinetic Energy of a Rolling Wheel

The Nature of a Wheel in This Section

Kinetic Energy of a Stationary Spinning Wheel

The Kinetic Energy of a Rolling Wheel

8.4.4 The Dynamics of a Soapbox Car

Defining the Soapbox Car

The Equation of Motion of a Soapbox Car

Choosing the Radius to Maximize the Rolling Speed

A Final Note: Looking at The Extreme Cases

Document: 8.5 Conic Curves

Movie:  8.5 Conic Curves

8.5.1 Introduction to Conic Curves

8.5.2 Rectangular Equations of Conic Curves

A Rectangular Equation of a Parabola

A Rectangular Equation of an Ellipse

A Rectangular Equation of a Hyperbola

Asymptotes of a Hyperbola

8.5.3 Exercises on Conic Curves

Exercise 1: A Parametric Form of the Equation of an Ellipse

Exercise 2: Adding the Distances from a Point on an Ellipse to the Focal Points

Exercise 3: A Parametric Form of the Equation of a Hyperbola

Exercise 4: Parametric Form of a Hyperbola Using Hyperbolic Functions

Exercise 5: Subtracting the Distances from a Point on an Ellipse to the Focal Points

Exercise 6: The Reflection Property of a Parabola

Exercise 7: The Reflection Property of an Ellipse

8.5.4 Polar Equations of Conic Curves

The Case $\varepsilon = 0$
The Case $\varepsilon > 0$

Overview of Chapter 9: Sequences and Series

Document: 9.1 Limits of Sequences

Movie:  9.1 Limits of Sequences

9.1.1 Introducing the Concepts

Sequences and Sequence Notation
Introducing Limits of Sequences
Convergent Sequences and Divergent Sequences
Illustrating Convergent and Divergent Sequences

9.1.2 Elementary Facts About Limits of Sequences

Limit of a Constant Sequence
Relating Limits and Inequalities
The Sandwich Rule for Sequences
An Analogue of the Sandwich Rule for Infinite Limits
The Arithmetical Rules for Limits

9.1.3 Some Exercises on Limits of Sequences

Exercise 1: The Limit $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$

Exercise 2: The Limit $\lim_{n \rightarrow \infty} (-1)^n$ Fails to Exist

Exercise 3: The Limit $\lim_{n \rightarrow \infty} \sqrt[n]{n}$

Exercise 4: The Limit $\lim_{n \rightarrow \infty} x^n$ when $x > 1$

Exercise 5: The Limit $\lim_{n \rightarrow \infty} x^n$ when $0 < x < 1$

Exercise 6: The Limit $\lim_{n \rightarrow \infty} x^n$ when $-1 < x < 1$

Exercise 7: The Limit $\lim_{n \rightarrow \infty} \frac{(-1)^n \log n}{n}$

Exercise 8: The Limit The Limit $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$

Exercise 9: The Limit $\lim_{n \rightarrow \infty} \frac{2^n}{n!}$

Exercise 10: The Limit $\lim_{n \rightarrow \infty} \frac{5^n}{n!}$

Exercise 11: The Limit $\lim_{n \rightarrow \infty} \frac{\log(n!)}{n^2}$

Exercise 12: The Important Limit $\lim_{n \rightarrow \infty} n \left(1 - \frac{n^p}{(n+1)^p}\right)$

9.1.4 Monotone Sequences

Introduction to Monotone Sequences
A Condition for an Increasing Sequence to Converge
A Final Note

Document: 9.2 An Intuitive Motivation of Infinite Series

Movie:  9.2 An Intuitive Motivation of Infinite Series

9.2.1 Our Objective in this Section

9.2.2 Some Examples to Illustrate Infinite Series

Example 1: The Sum $0 + 0 + 0 + 0 + 0 + 0 + \dots$

Example 2: The Sum $1 + 1 + 1 + 1 + 1 + 1 + \dots$

Example 3: Taking $a_n = \begin{cases} 1 & \text{if } 1 \leq n \leq 4 \\ 0 & \text{if } n \geq 5 \end{cases}$

Example 4: The Infinitely Repeating Decimal $0.\bar{1}$

Example 5: The Infinitely Repeating Decimal $0.\bar{9}$

Example 6: The Infinitely Repeating Decimal $0.\overline{473}$

Example 7: The Sum $1 + x + x^2 + x^3 + \dots$ When $-1 < x < 1$

Example 8: The Sum $1 - x + x^2 - x^3 + x^4 - \dots$ When $-1 < x < 1$

Example 9: The Sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

Example 10: The sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Example 11: The sum $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

Example 12: The sum $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

Example 13: The Equation $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Example 14: The Equation $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Example 15: The Equation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

Example 16: Comparing the Series Expansions of exp, cos, and sin

Example 17: The Equation $x^2 = \frac{\pi^2}{3} - \frac{4\cos x}{1^2} + \frac{4\cos 2x}{2^2} - \frac{4\cos 3x}{3^2} + \dots$

9.2.3 Concluding Remarks

Document: 9.3 Introduction to Infinite Series

Movie:  9.3 Introduction to Infinite Series

9.3.1 The Series with n th Term a_n

9.3.2 Convergence and Divergence of Series

9.3.3 Some Examples to Illustrate the Idea of a Series

Example 1: The Series $\sum 0$

Example 2: The Series $\sum 1$

Example 3: Taking $a_n = \begin{cases} 1 & \text{if } 1 \leq n \leq 4 \\ 0 & \text{if } n \geq 5 \end{cases}$

Example 4: The Series $\sum \frac{1}{n(n+1)}$

Example 5: The Series $\sum \frac{2}{n(n+1)(n+2)}$ for Each n

Example 6: The Geometric Series $\sum x^{n-1}$

Example 7: The Series $\sum \log\left(1 + \frac{1}{n}\right)$

Example 8: The Series $\sum (-1)^{n-1}$

9.3.4 The n th Term Criterion for Divergence

Introduction to the n th Term Criterion for Divergence

Proof of the n th Term Criterion for Divergence

9.3.5 A Return to the Examples of 9.3.3

Example 1: The Series $\sum 0$

Example 2: The Series $\sum 1$

Example 3: Taking $a_n = \begin{cases} 1 & \text{if } 1 \leq n \leq 4 \\ 0 & \text{if } n \geq 5 \end{cases}$

Example 4: The Series $\sum \frac{1}{n(n+1)}$

Example 5: The Series $\sum \frac{2}{n(n+1)(n+2)}$

Example 6: The Geometric Series $\sum x^{n-1}$

Example 7: The Series $\sum \log\left(1 + \frac{1}{n}\right)$

Example 8: The Series $\sum (-1)^{n-1}$

9.3.6 Some Applications of the n th Term Criterion for Divergence

A Ratio Criterion for Divergence

Testing the Series $\sum \frac{n!}{6^n}$

Divergence of the Series $\sum \frac{(2n)!}{(n!)^2}$

Divergence of The Series $\sum \frac{(-1)^n 4^n (n!)^2}{(2n)!}$

A Problem that We Cannot Solve Right Now: Test the Series $\sum \frac{(2n)!}{4^n (n!)^2}$

A Limit Form of the Ratio Criterion for Divergence

Divergence of the Series $\sum \frac{3^n}{n^{10}}$

Divergence of the Series $\sum \frac{(3^n)(n!)}{n^n}$

9.3.7 A Quick Summary of What We Know at Present

Document: 9.4 Convergence of Nonnegative Series

Movie:  9.4 Convergence of Nonnegative Series

9.4.1 Introduction to Nonnegative Series

9.4.2 The Integral Comparison Test

Divergence of the Series $\sum \frac{1}{n}$

Convergence of the Series $\sum \frac{1}{n^2}$

The General Form of the Integral Comparison Test

The p -Series

The p -Series When $p > 1$

The p -Series When $p < 1$

Conclusion: Convergence Criteria for the p -Series

A Sharper Form of the p -Series

The Case $p = 1$

The Case $p < 1$

The Case $p > 1$

9.4.3 Optional: A Sharper Type of Integral Comparison

An Extension of the Integral Comparison Test

Euler's Constant

The Limit $\lim_{n \rightarrow \infty} \sum_{j=n+1}^{2n} \frac{1}{j}$

Summing the Series $\sum \frac{(-1)^{n-1}}{n}$

Summing the Series $\sum \frac{1}{n(2n-1)}$

9.4.4 Comparing Series with One Another

The Comparison Test: Inequality Form

The Comparison Test: Limit Form

9.4.5 Some Exercises on The Comparison Test

Exercise 1: Testing the Series $\sum \frac{\sin^2 n}{n^2}$

Exercise 2: An Unsuccessful Attempt to Test the Series $\sum \frac{\sin^2 n}{n}$

Exercise 3: Testing the Series $\sum \frac{n}{n^4 + 7}$

Exercise 4: Testing the Series $\sum \frac{n}{n^4 - 7}$

Exercise 5: Testing the Series $\sum \frac{1}{n^{3/2} + n}$

Exercise 6: Testing the Series $\sum \frac{1}{n^{3/2} - n}$

Exercise 7: Testing the Series $\sum \frac{n}{\sqrt{n^4 - n^2 + 2}}$

Exercise 8: Testing the Series $\sum \frac{\log n}{n^2}$

Exercise 9: Testing the Series $\sum \frac{n \log n}{\sqrt{n^5 - n^2 + 2}}$

Exercise 10: Testing the Series $\sum \frac{1}{n^{1+1/n}}$

Exercise 11: Testing the Series $\sum \frac{1}{n^{1+(\log n)/n}}$

Exercise 12: Testing the Series $\sum \frac{1}{n^{1+(\log n)^2/n}}$

Exercise 13: Testing the Series $\sum \left(\frac{n}{n+1}\right)^n$

Exercise 14: Testing the Series $\sum \left(\frac{1}{\log n}\right)^3$

Exercise 15: Testing the Series $\sum \left(\frac{1}{\log n}\right)^n$

Exercise 16: Testing the Series $\sum \left(\frac{1}{\log n}\right)^{\log n}$

Exercise 17: Testing the Series $\sum \left(\frac{1}{\log \log n}\right)^{\log n}$

Exercise 18: Testing the Series $\sum \left(\frac{1}{\log n}\right)^{\log \log n}$

9.4.6 The Elementary Ratio Tests

Introducing the Ratio Tests

The Ratio Comparison Test

The d'Alembert Ratio Test, Inequality Form

The d'Alembert Ratio Test, Limit Form, Often Known as "The Ratio Test"

9.4.7 Some Exercises that Rely on d'Alembert's Test (Exercises on "The Ratio Test")

Exercise 1: Testing the Series $\sum \frac{n^{1000000}}{2^n}$

Exercise 2: Testing the Series $\sum \frac{2^n}{n!}$

Exercise 3: Testing the Series $\sum \frac{n!}{n^n}$

Exercise 4: Testing the Series $\sum \frac{n^{cn}}{n!}$ Given $c < 1$

Exercise 5: Testing the Series $\sum \frac{(2^n)(n!)}{n^n}$

Exercise 6: Testing the Series $\sum \frac{(3^n)(n!)}{n^n}$

Exercise 7: An Unsuccessful Attempt to Test the Series $\sum \frac{(e^n)(n!)}{n^n}$

Exercise 8: An Unsuccessful Attempt to Test the Series $\sum \frac{n^n}{(e^n)(n!)}$

Exercise 9: Testing the Series $\sum \frac{(2n)!}{5^n(n!)^2}$

Exercise 10: Testing the Series $\sum \frac{(2n)!}{3^n(n!)^2}$

Exercise 11: Testing the Series $\sum \frac{4^n(n!)^2}{(2n)!}$

Exercise 12: An Unsuccessful Attempt to Test the series $\sum \frac{(2n)!}{4^n(n!)^2}$

Exercise 13: Testing the Series $\sum \frac{((2n)!)^3}{((3n)!)^2}$

Exercise 14: Testing the Series $\sum \frac{(\log n)^n}{c^n(\log 2)(\log 3)\cdots(\log n)}$ for $c > 0$

Exercise 15: A Second Visit to the Series $\sum \frac{(e^n)(n!)}{n^n}$

9.4.8 The More Powerful Ratio Tests

Introduction to the More Powerful Tests
The Inequality Form of Raabe's Ratio Test
The Limit Form of Raabe's Test
A Level Two Ratio Test
A Level Three Ratio Test

9.4.9 Some Exercises on the More Powerful Ratio Tests

Exercise 1: Successful Testing of the Series $\sum \frac{(2n)!}{4^n(n!)^2}$

Exercise 2: Testing the Series $\sum \frac{|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)|}{n!}$

Exercise 3: Successful Testing of the Series $\sum \frac{n^n}{(e^n)(n!)}$

Exercise 4: Testing the Series $\sum \left(\frac{(2n)!}{4^n(n!)^2}\right)^2$

Document: 9.5 Absolute and Conditional Convergence

Movie:



9.5 Absolute and Conditional Convergence

9.5.1 Introduction to Convergence of Series Whose Terms Can Change Sign

9.5.2 Absolutely Convergent Series

Definition of an Absolutely Convergent Series
Convergence of Absolutely Convergent Series
Some Examples of Absolutely Convergent Series

9.5.3 Conditionally Convergent Series

9.5.4 The Alternating Series Test

Statement of the Alternating Series Test
Warning: Read the Statement of the Alternating Series Test Carefully!
Some Examples of Series Whose Conditional Convergence Can be Deduced from the Alternating Series Test
Proof of the Alternating Series Test
An Error Estimate for Alternating Series
Approximations to $\log 2$

9.5.5 Dirichlet's Test (Optional)

Statement of Dirichlet's Test
Proof of Dirichlet's Test
An Error Estimate for a Series Tested by Dirichlet's Test

9.5.6 Some Exercises on Dirichlet's Test (Optional)

Exercise 1: Testing the Series $\sum \frac{\sin nx}{n}$

Exercise 2: Testing the Series $\sum \frac{\cos nx}{n}$

Exercise 3: Testing the Series $\sum \frac{\cos^2 nx}{n}$

Exercise 4: Testing the Series $\sum \frac{\sin^2 nx}{n}$

Exercise 5: Conditional Convergence of $\sum \frac{\sin nx}{n}$ and $\sum \frac{\cos nx}{n}$

Exercise 6: A Relationship Between $\sum a_n$ and $\sum a_n^3$

9.5.7 Some Further Series to Test with the Alternating Series Test or Dirichlet's Test (Optional)

Introducing This Topic
A Special Technique for Testing Alternating Series
Testing the Series $\sum \frac{(-1)^n(2n)!}{4^n(n!)^2}$

Testing the Series $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$

Testing the Series $\sum \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{n}\right)(-1)^n$

Testing the Series $\sum \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)(-1)^n$

Document: 9.6 Power Series

Movie:



9.6 Power Series

9.6.1 Introduction to Power Series

9.6.2 Some Examples of Power Series

Example 1: The Geometric Series $\sum x^n$

Example 2: The Series $\sum \frac{x^n}{n!}$

Example 3: The Series $\sum \frac{(-1)^{n-1} x^n}{n}$

Example 4: The Series $\sum \frac{(-1)^{n-1} (x-4)^{2n-1}}{2n-1}$

Example 5: The Series $\sum n(x+3)^{n-1}$

Example 6: The Series $\sum (n!)x^n$

9.6.3 Radius and Interval of Convergence of a Power Series

The Case $0 < r < \infty$

The Case $r = 0$

The Case $r = \infty$

9.6.4 Some Exercises on Radius and Interval of Convergence

Exercise 1: The Series $\sum \frac{(x-5)^n}{2^n n^2}$

Exercise 2: The Series $\sum \frac{(x-5)^n}{2^n n}$

Exercise 3: The Series $\sum \frac{(-1)^n (x-5)^n}{2^n n}$

Exercise 4: The Series $\sum \frac{n(x-5)^n}{2^n}$

Exercise 5: The Series $\sum \frac{(x-5)^{2n}}{n 3^n}$

Exercise 6: The Series $\sum \frac{(n!)^2}{(2n)!} x^n$

Exercise 7: The Series $\sum \frac{(2n)!}{(n!)^2} x^n$

Exercise 8: The Binomial Series $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n$

9.6.5 The Principal Facts About Power Series

The Derivative of the Sum of a Power Series

Higher Derivatives of the Sum of a Power Series

A Formula for the Coefficients of a Power Series

The Taylor and Maclaurin Series of a Given Function

9.6.6 Some Important Examples of Taylor Series

Example 1: The Geometric Series

Example 2: The Alternating Geometric Series

Example 3: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \log(1+x)$, Found by the Derivative Method

Example 4: The Equation $\sum_{n=0}^{\infty} \frac{(1)^n}{2n+1} x^{2n+1} = \arctan x$, Found by the Derivative Method

Example 5: The Equation $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, Found by the Derivative Method

Example 6: The Equation $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, Found by the Remainder Method

Example 7: The Equations $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ Found by the Derivative Method

Example 8: The Equation $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ Found by the Remainder Method

Example 9: The Equation $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ Found by the Remainder Method

Example 10: A Bump Function

9.6.7 The Binomial Expansion

An Introduction to the Binomial Series
The Binomial Coefficients
A Needed Fact About the Binomial Coefficients
A Needed Fact About the Sum of the Binomial Series
Summing the Binomial Series

9.6.8 Abel's Theorem

9.6.9 Some Applications of Abel's Theorem

Example 1: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \log 2$

Example 2: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

Example 3: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} = \frac{1}{\sqrt{2}}$

9.6.10 Tauber's Theorem

Overview of Chapter 10: Some Basics in Linear Algebra

Document: 10.1 A Glance at Second and Third Order Determinants

Movie:  10.1 A Quick Look at Second and Third Order Determinants

10.1.1 Second Order Determinants


Definition of a Second Order Determinant
Example 1
Example 2
Solving Two Equations for Two Unknowns

10.1.2 Third Order Determinants

Definition of a Third Order Determinant
Alternative Expansions of a Third Order Determinant
Example of a Third Order Determinant
Solving Three Equations for Three Unknowns

10.1.3 More General Determinants

Document: 10.2 Vectors in Space

Movie:  10.2 Vectors in Space

10.2.1 Preliminary Note

10.2.2 Introducing the Arithmetical Operations in \mathbf{R}^n

Definition of the Space \mathbf{R}^n
Addition and Subtraction in \mathbf{R}^n

10.2.3 Some Properties of Addition and Subtraction in \mathbf{R}^n

Adding any Point to the Origin
The Commutative Law for Addition in \mathbf{R}^n
The Associative Law for Addition in \mathbf{R}^n
Some Facts About Subtraction
The Symbol $-A$

10.2.4 Scalar Multiplication in \mathbf{R}^n

Introducing Scalar Multiplication
Definition of Scalar Multiplication in \mathbf{R}^n
Some Properties of Scalar Multiplication in \mathbf{R}^n

10.2.5 Linear Combinations

Definition of a Linear Combination
Two Examples of Linear Combinations
 Example 1
 Example 2
The Standard Basis in \mathbf{R}^n
 The Standard Basis in \mathbf{R}^2
 The Standard Basis in \mathbf{R}^3
 Extending the Idea of Standard Basis to \mathbf{R}^n

10.2.6 Geometric Interpretation of the Arithmetical Operations in \mathbf{R}^2 and \mathbf{R}^3

Norm of a Point in \mathbf{R}^2
Using the Norm to Find the Length of a Line Segment in \mathbf{R}^2
Coordinate Axes and the Norm in \mathbf{R}^3
Using the Norm to Find the Length of a Line Segment in \mathbf{R}^3
Line Segments with the Same Length and Direction
The Parallelogram Rule for Addition
Line Segments with the Same Direction and Different Lengths
Line Segments with Opposite Directions
Norm of a Point in \mathbf{R}^n : Definition of a Unit Vector
Some Simple Facts About the Norm in \mathbf{R}^n
 The Norm is Zero only at O
 The Norm and Scalar Multiplication
 Dividing a Point by its Norm to Produce a Unit Vector

10.2.7 Exercises on the Geometric Interpretation of the Arithmetical Operations in \mathbf{R}^2 and \mathbf{R}^3

Exercise 1: Midpoint of a Line Segment
Exercise 2: An Application to Geometry
Exercise 3: An Application to Geometry
Exercise 4: An Application to Geometry
Exercise 5: A 3D Analogue of Exercise 4

10.2.8 The Concept of a Vector

Motivating the Vector Concept by Looking at Forces that Act on a Particle
Introducing the Concept of a Vector
Another Look at Vector Addition

10.2.9 The Inner Product (Dot Product)

Preliminary Discussion of the Inner Product (Dot Product)
Definition of the Inner Product
The Inner Product of a Point with Itself
The Commutative Law for the Inner Product
The Inner Product and Scalar Multiplication

The Distributive Law for the Inner Product
 Inner Product of Points with Norm One
 The Cauchy-Schwarz Inequality
 The Minkowski Inequality
 The Triangle Inequality
 A Geometric Interpretation of the Inner Product in \mathbf{R}^2 and \mathbf{R}^3
 Perpendicular Line Segments in \mathbf{R}^2 and \mathbf{R}^3
 Orthogonality in \mathbf{R}^n
 Orthonormal Sets
 Expressing Any Vector in Terms of an Orthonormal Set

10.2.10 Some Exercises on the Inner Product

Exercise 1: Finding an Angle
 Exercise 2: The set $\{(\cos \theta, \sin \theta), (-\sin \theta, \cos \theta)\}$ is orthonormal.
 Exercise 3: $(\cos \alpha, \sin \alpha) \cdot (\cos \beta, \sin \beta)$
 Exercise 4: Angle in a Semicircle
 Exercise 5: Diagonals of a Rhombus
 Exercise 6: Diagonals of a Rectangle
 Exercise 7: Altitudes of a Triangle
 Exercise 8: The Euler Line of a Triangle

10.2.11 The Cross Product in \mathbf{R}^3

Definition of the Cross Product in \mathbf{R}^3
 Some Examples of Cross Products
 Example 1
 Example 2
 The Equation $A \times A = O$
 The Equation $A \times B = -B \times A$
 The Distributive Law for the Cross Product
 The Cross Product and Scalar Multiplication
 Failure of the Associative Law
 The Scalar Triple Product
 The Vector Triple Product
 The Norm of a Cross Product
 The Direction of $A \times B$

10.2.12 Some Exercises on Cross Products

Exercise 1: An Application to Area of a Triangle
 Exercise 2: An Application to Area of a Triangle
 Exercise 3: Finding the Area of a Given Triangle
 Exercise 4: An Exercise on Triple Products
 Exercise 5: Another Exercise on Triple Products
 Exercise 6: Another Exercise on Triple Products

10.2.13 Volume of a Parallelepiped

Document: 10.3 Lines and Planes in \mathbf{R}^3

Movie:  10.3 Lines and Planes in \mathbf{R}^3

10.3.1 Lines and Parametric Lines in \mathbf{R}^2

Introduction to This Section
 Straight Line Graphs of the type $ax + by = d$ in \mathbf{R}^2
 Parametric Form of the Equation of a Straight Line in \mathbf{R}^2

10.3.2 Some Exercises on Lines in \mathbf{R}^2

Exercise 1: Finding The Intersection of Two Lines
 Exercise 2: Finding The Intersection of Two Parametric Lines
 Exercise 3: A Line Perpendicular to Given Direction
 Exercise 4: Dropping a Perpendicular to a Line
 Exercise 5: Dropping a Perpendicular to a Parametric Line

10.3.3 Lines and Planes in R^3

The Two Kinds of Equation
The Equation of a Plane
Parametric Equations of a Line

10.3.4 Exercises on Lines and Planes

Exercise 1: Equation of a Plane Containing a Given Point and Perpendicular to a Given Direction
Exercise 2: Equation of a Plane Containing a Given Point and Perpendicular to a Given Line Segment
Exercise 3: Equation of a Line Containing a Given Point and Parallel to a Given Line
Exercise 4: Equation of a Line Containing Two Given Points
Exercise 5: Intersection of a Line and a Plane
Exercise 6: Failure of Intersection of a Line and a Plane
Exercise 7: Intersection of Two Lines
Exercise 8: Angle Between Two Given Lines
Exercise 9: Plane Containing Two Given Lines
Exercise 10: Plane Containing Three Given Points
Exercise 11: Plane Containing a Line and a Point
Exercise 12: Line Perpendicular to Two Given Lines
Exercise 13: Dropping a Perpendicular to a Line
Exercise 14: Point in a Line Closest to a Given Point
Exercise 15: Common Perpendicular Between Two Lines
Exercise 16: Perpendicular from a Point to a Plane

10.3.5 Parametric Equation of a Plane in R^3

Introducing the Parametric Equation of a Plane
An Example of a Parametric Equation of a Plane

Overview of Chapter 11: Multivariable Differential Calculus

Document: 11.1 Surfaces and Curves in R^3

Movie:  11.1 Surfaces and Curves in R^3

11.1.1 Preliminary Note on This Section

11.1.2 Surfaces as Implicit Plots and Parametric Surfaces

11.1.3 Some Examples of Surfaces

Example 1: Plotting a Cone
Example 2: Plotting a Circular Paraboloid
Example 3: Plotting an Ellipsoid
Example 4: Plotting a Cone and a Hemisphere
Example 5: Plotting an Hyperboloid of One Sheet
Example 6: Plotting an Hyperboloid of Two Sheets
Example 7: Plotting a Corkscrew
Example 8: Plotting A Double Sea Shell
Example 9: Plotting a Cylinder
Example 10: Plotting Möbius Band
Example 11: Plotting a Cylinder with Two Twists
Example 12: Plotting a Cylinder with Three Twists
Example 13: Plotting a Cylinder with Four Twists
Example 14: Twisting a Cylinder
Example 15: A Surface with a Surprise

11.1.4 Parametric Curves

Motivating the Idea of a Parametric Curve in R^3
11.1.4.2 Definition of a Parametric Curve in R^3

11.1.5 Some Examples of Curves

- Example 1: Plotting a Spiral on a Cylinder
- Example 2: Plotting a Spiral on a Cone
- Example 3: Plotting an Exponential Spiral
- Example 4: Plotting Two Interlocking Closed Curves
- Example 5: Plotting the Hardy-Walker Knotted Closed Curve

Document: 11.2 The Calculus of Curves

Movie: 11.2 The Calculus of Curves

11.2.1 Limits and Continuity of Parametric Curves

- Limit of a Parametric Curve at a Given Number
- Continuity of a Curve

11.2.2 Some Examples to Illustrate Limits and Continuity of Curves

Example 1: $\lim_{t \rightarrow 3} (2t - 3, t^2, 5t)$

Example 2: $\lim_{t \rightarrow 3} (2t - 3, t^2, 5t)$

Example 3: A Discontinuous Curve

11.2.3 Velocity, also called the Derivative of a Curve

- Definition of the Velocity of a Curve
- Speed of a Curve
- Acceleration of a Curve
- An Example to Illustrate the Velocity, Speed and Acceleration of a Curve

11.2.4 Geometric Interpretation of Velocity and Speed

- The Direction of the Velocity of a Curve
- Using Speed to Find the Length of a Curve

11.2.5 Some Exercises on Velocity and Speed

- Exercise 1: Length of a Curve
- Exercise 2: Length of a Curve
- Exercise 3: A Product Rule for Scalar Multiplication
- Exercise 4: A Sum Rule
- Exercise 5: A Product Rule for the Dot Product
- Exercise 6: A Product Rule for the Cross Product
- Exercise 7: Curves with Constant Norm
- Exercise 8: The Equation $\frac{d}{dt} P(t) \times P'(t) = P(t) \times P''(t)$

11.2.6 Curvature, Principal Normal, Binormal, and Torsion of a Curve

- Velocity of a Curve Whose Norm is Constant
- Unit Tangent Vector of a Parametric Curve
- Principal Normal of a Parametric Curve
- The Curvature of a Parametric Curve
- The Equation $T'(t) = k(t)s'(t)N(t)$
- The Curvature of a Circle is the Reciprocal of Its Radius
- Center of Curvature and Evolute of a Parametric Curve
- The Binormal of a Parametric Curve
- The Orthonormal Triple $\{T(t), N(t), B(t)\}$
- The Torsion of a Parametric Curve
- The Frenet Formulas

11.2.7 The Acceleration of a Parametric Curve

- Definition of Acceleration of a Parametric Curve
- The Relationship Between Acceleration, Curvature and Principal Normal
- The Product $P'(t) \times P''(t)$ and a Useful Formula for $k(t)$

11.2.8 Some Exercises on Curvature

- Exercise 1: Working with $P(t) = (e^t \cos t, e^t \sin t, e^t)$

- Exercise 2: Working with $y = x^2$
- Exercise 3: Working with $y = f(x)$
- Exercise 4: An Animation Showing the Evolute of a Cycloid
- Exercise 5 Animating the Evolute of a Four Leaf Rose

11.2.9 Motion of a Particle in Space: Newton's Law

- The Basic Definitions
- Newton's Law
- Expressing the Force Acting on a Particle in Terms of Curvature

11.2.10 Planetary Motion

- Introduction to Planetary Motion
- Some Technical Preliminaries
 - The Identity $f(\theta)(\cos \theta, \sin \theta) = g(\theta)(-\sin \theta, \cos \theta)$
 - The Equation $f''(x) + f(x) = 0$
 - The Equation $f''(x) + f(x) = c$
 - An Alternative Form of the Solution
- An Analysis of Planetary Motion

Document: 11.3 Real Valued Functions

Movie:  11.3 Real Valued Functions

11.3.1 Introduction to Real Valued Functions

11.3.2 Some Examples of Real Valued Functions

- Example 1: $f(x, y, z) = x + ye^{xz}$
- Example 2: $f(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}$
- Example 3: $f(u, v, w, x, y) = \frac{yx \sin(vy) + \log(u^2 + v^2)}{\sqrt{1 + u^2 + v^2 + w^2 + x^2 + y^2}}$
- Example 4: $f(x, y) = \frac{(x^2 - y^2)^2}{x^2 + y^2}$
- Example 5: $f(x, y) = (\sin x - \sin y)^2$
- Example 6: $f(x, y) = \frac{x \sin y - y \sin x}{x^2 + y^2}$

11.3.3 Limits of Real Valued Functions

- Closeness in the Space \mathbf{R}^2
- Closeness in the Space \mathbf{R}^3
- Limit at a Given Point in \mathbf{R}^2
- Limit at a Given Point in \mathbf{R}^3

11.3.4 Some Examples of Limits

- Example 1: $\lim_{(x,y) \rightarrow (0,0)} (x^2 + 3xy - 2y^2) = 0$
- Example 2: $\lim_{(x,y) \rightarrow (-1,2)} (x^2 + 3xy - 2y^2) = -13$
- Example 3: $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1$
- Example 4: $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$
- Example 5: $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2}$
- Example 6: $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$
- Example 7: $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$ Does not Exist
- Example 8: An Example of a Repeated Limit

Document: 11.4 Partial Derivatives

Movie:



11.4 Partial Derivatives

11.4.1 Introduction to Partial Derivatives

- Partial Derivatives of a Function of Two Variables
- Functions of More than Two Variables
- A Geometric Interpretation of Partial Derivatives
- A More Precise Approach to Partial Derivatives
- Higher Order Partial Derivatives
- Equality of Second Order Mixed Partial Derivatives

11.4.2 Some Exercises on Partial Derivatives

- Exercise 1: Working out Partial Derivatives
- Exercise 2: Obtaining a Relationship among Partial Derivatives
- Exercise 3: Obtaining a Relationship among Partial Derivatives
- Exercise 4: Obtaining a Relationship among Partial Derivatives
- Exercise 5: Obtaining a Relationship among Partial Derivatives
- Exercise 6: Obtaining the Laplace Equation
- Exercise 7: Obtaining the Laplace Equation
- Exercise 8: The Cauchy-Riemann and Laplace Equations
- Exercise 9: Failure of Equality of Mixed Second Order Partial Derivatives

11.4.3 The Chain Rule

- An Example to Motivate the Chain Rule
- A Second Example to Motivate the Chain Rule
- The Chain Rule for Functions of Two Variables
- The Chain Rule for Functions of Three Variables
- The Chain Rule for Functions of n Variables

11.4.4 Some Exercises on the Chain Rule

- Exercise 1: Illustrating the Chain Rule
- Exercise 2: Illustrating the Chain Rule
- Exercise 3: Changing to Polars
- Exercise 4: A Linear Transformation
- Exercise 5: Applying the Chain Rule to the Second Derivative
- Exercise 6: Changing to Polars, Second Derivatives
- Exercise 7: Changing to Sphericals, Second Derivatives
- Exercise 8: Euler's Formula for Homogeneous Functions

Document: 11.5 Vector Fields

Movie:



11.5 Vector Fields

11.5.1 Introduction to Vector Fields

- The Force of Gravity as a Vector Field
- Velocity of a Flowing Fluid as a Vector Field
- Definition of a Vector Field
- Scalar Fields

11.5.2 Some Examples of Vector Fields

- Example 1: Plotting a Vector Field
- Example 2: Plotting a Vector Field
- Example 3: Plotting a Vector Field
- Example 4: Plotting a Vector Field

11.5.3 Gradient, Divergence, Laplacian, and Curl

- Gradient of a Real Function
- Gradient of a Real Function
- The Laplacian
- The Curl of a Vector Field

The Operator ∇ Called Nabla or Del

11.5.4 Exercises on Gradient, Curl, and Divergence

Exercise 1: $\text{curl}(\text{grad}(x^2 \sin(y^2 + z^3))) = (0, 0, 0)$

Exercise 2: $\text{curl}(xy, yz, zx)$

Exercise 3: $\text{grad}\left(\frac{mk}{\sqrt{x^2 + y^2 + z^2}}\right)$

Exercise 4: Finding v or which $\nabla v(x, y, z) = (yz, zx, xy)$

Exercise 5: $\text{curl}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

Exercise 6: $\text{div curl } F = 0$

Exercise 7: $\text{grad}f(x^2 + y^2 + z^2)$

Exercise 8: The Chain Rule Using a Dot Product

11.5.5 Conservative Vector Fields and Potential of a Field

Potential of a Vector Field

A Necessary Condition a Vector Field to be Conservative

11.5.6 Exercises on Conservative Fields and Potential

Exercise 1: A Non Conservative Field

Exercise 2: Finding a Potential for a Given Field

Exercise 3: Finding a Potential for a Given Field

Exercise 4: A Non Conservative Field

Exercise 5: Finding a Potential for a Given Field

11.5.7 Directional Derivative

Motivating the Idea of a Directional Derivative

Definition of the Directional Derivative of a Scalar Field

A Useful Formula for a Directional Derivative

Choosing the Direction to Maximize the Directional Derivative

11.5.8 Exercises on Directional Derivatives

Exercise 1: Finding a Directional Derivative

Exercise 2: Direction of Maximum Increase of a Function

Exercise 3: Direction of Maximum Decrease of a Function

Document: 11.6 Further Topics on Partial Differentiation

Movie:



11.6 Further Topics on Partial Differentiation

11.6.1 A Quick Look at Matrix Arithmetic

Notation for Matrices

Addition and Subtraction of Matrices

Multiplication of a Matrix by a Number

Multiplication of One Matrix by Another

The Identity Matrix

Invertible and Singular Matrices

A Relationship Between Matrix Multiplication and Determinants

11.6.2 Some Exercises on Matrix Arithmetic

Exercise 1: Working out a Simple Product

Exercise 2: Working out a Simple Product

Exercise 3: Product of Invertible Matrices

Exercise 4: A System of Linear Equations in Matrix Form

Exercise 5: Solving a System of Linear Equations Using Matrix Notation

11.6.3 The Jacobian Matrix of a Vector Field

Writing the Coordinates of a Vector Field Vertically

Motivating the Idea of a Jacobian Matrix

The Jacobian Matrix of a Vector Field in \mathbb{R}^3

The Jacobian Matrix of a Function from a Region in \mathbf{R}^6 into \mathbf{R}^4
The General Case of a Jacobian Matrix

11.6.4 Expressing the Chain Rule in Matrix Form

A Simple Example Showing the Chain Rule in Matrix Form
Revisiting the Chain Rule for Real Functions
The $4 \times 2 \times 3$ Form of the Chain Rule
The General $n \times m \times k$ Form of the Chain Rule


11.6.5 Implicit Differentiation

A Review of Implicit Differentiation as We Saw It in Section 3.8
Applying Implicit Differentiation to a Single Equation in Three Unknowns
Applying Implicit Differentiation to Two Equations in Three Unknowns: A Special Case
Applying Implicit Differentiation to Two Equations in Three Unknowns: The General Case
Applying Implicit Differentiation to Four Equations in Seven Unknowns:
The General Implicit Differentiation Problem

11.6.6 Principal Normal of a Parametric Surface

Introducing the Concept of Principal Normal
Principal Normal of a Sphere
Principal Normal of a Cone
Finding a Normal to a Surface of the Form $f(x, y, z) = 0$
Tangent Plane to the Surface $x^2y + yz^2 = 20$ at $(1, 2, 3)$
Tangent Plane to the Surface $x^3 + y^3 + z^3 + 3xyz = 6$ at $(1, 1, 1)$
Tangent Plane to the Surface $ze^{xy} - 4x^2 - 4y^2 = e - 8$ at $(1, 1, 1)$
Tangent Plane to the Surface $e^{-x^2-y^2-z^2}(4x^2 + 5xyz + 4y^2 + 4z^2) = 17e^{-3}$ at $(1, 1, 1)$

Document: 11.7 Maxima and Minima

Movie:  11.7 Maxima and Minima

11.7.1 Definitions of Maxima and Minima

Definition of Maximum and Minimum of a Function
Definition of Local Maximum and Local Minimum of a Function

11.7.2 Some Examples to Illustrate the Definitions

Example 1: Illustrating Maxima and Minima
Example 2: Illustrating Maxima and Minima
Example 3: Illustrating Maxima and Minima
Example 4: Illustrating Maxima and Minima

11.7.3 Basic Facts About Maxima and Minima

Existence of Maxima and Minima of a Function
Fermat's Theorem
Critical Points of a Function
Finding Maxima and Minima of a Given Function
Saddle Points
The Second Derivative Test for Maxima and Minima

11.7.4 Exercises on Maxima and Minima


Exercise 1: Maximum and Minimum of a Polynomial
Exercise 2: Maximum and Minimum of a Polynomial on a Disk
Exercise 3: A Monkey Saddle
Exercise 4: Finding Critical Points
Exercise 5: A Box Problem
Exercise 6: A Maximum Minimum Problem that Requires a Computer Algebra System

11.7.5 The Standard Simplex in \mathbf{R}^n

The Standard Simplex in \mathbf{R}^1 , \mathbf{R}^2 , and \mathbf{R}^3
Definition of the Standard Simplex Q^n
A Maximum Minimum Problem on the Simplex Q^n

Overview of Chapter 12: Multivariable Integral Calculus

Document: 12.1 Integration on Curves

Movie:  12.1 Integration on Curves

12.1.1 Integration on a Smooth Curve

Definition of a Smooth Curve

Integrals of the Type $\int_p f dx$, $\int_p f dy$, and $\int_p f dz$

Integrals of the Type $\int_p F \cdot dP = \int_p F \cdot (dx, dy, dz) = \int_p f dx + g dy + h dz$

Application to Work Done by a Force

12.1.2 Examples of Integrals on Smooth Curves

Example 1

Example 2

Example 3

12.1.3 Fundamental Theorem of Calculus for Integrals on Curves

Introduction to the Fundamental Theorem

Statement of the Fundamental Theorem for Integrals of the Type $\int_p F \cdot dP$

Path Independence and the Fundamental Theorem

The Role of “Whirlpools”

12.1.4 Exercises on Integrals on Curves

Exercise 1: Evaluating an Integral on a Curve

Exercise 2: Integral on a Straight Line Segment

Exercise 3: Integrating a Conservative Field on an Unknown Curve

Exercise 4: Integrating a Conservative Field on an Unknown Curve

Exercise 5: Integrating a Non Conservative Field

Exercise 6: The Potential of the Force of Gravity

12.1.5 Reparametrizing a Curve

Motivating the Idea of a Reparametrization of a Curve

Reparametrizing a Curve in the Direction of Travel

Reparametrizing a Curve Reversing the Direction of Travel

An Animation to Illustrate a Reparametrization that Reverses the Direction of Travel

Integrating on a Reparametrization that is in the Direction of Travel

Integrating on a Reparametrization that Reverses the Direction of Travel

12.1.6 Integration on a Chain of Smooth Curves

Motivating the Idea of a Chain of Curves

Definition of a Chain of Curves

Integrating on a Chain of Curves

Integrating around a Triangle

12.1.7 Exercises on Integrals on Chains

Exercise 1: Evaluating an Integral on a Chain

Exercise 2: Integrating Around a Square

Exercise 3: An Integral Around a Triangle

12.1.8 A More General Notion of a Chain of Curves

Document: 12.2 Integration of a Function of Two Variables

Movie:  12.2 Integration of a Function of Two Variables

12.2.1 *Iterated Integrals in Two Variables*

Iterated Integrals with Constant Limits
More General Iterated Integrals

12.2.2 *Some Examples of Iterated Integrals*

Example 1: Evaluating an Iterated Integral
Example 2: Evaluating an Iterated Integral
Example 3: Evaluating an Iterated Integral
Example 4: Evaluating an Iterated Integral
Example 5: Evaluating an Iterated Integral
Example 6: Evaluating an Iterated Integral
Example 7: Evaluating an Iterated Integral
Example 8: Some Meaningless Iterated Integrals

12.2.3 *The Fichtenholz Theorem*

Note to Instructors on the Fichtenholz Theorem
Introduction to Fichtenholz Theorem
Statement of the Fichtenholz Theorem

12.2.4 *Some Exercises on Iterated Integrals*

Exercise 1: Inverting the Order of an Iterated Integral
Exercise 2: Inverting the Order of an Iterated Integral
Exercise 3: Inverting the Order of an Iterated Integral
Exercise 4: Inverting the Order of an Iterated Integral
Exercise 5: Inverting the Order of an Iterated Integral
Exercise 6: Evaluating the Integral $\int_0^{\infty} e^{-x^2} dx$
Exercise 7: Failure of Equality of Iterated Integrals
Exercise 8: Failure of Equality of Iterated Integrals

12.2.5 *Introduction to Integration over Regions*

12.2.6 *Integrals over Regions in \mathbf{R}^1*

Integral over an Interval [a,b] in \mathbf{R}^1
The General Case of a Region in \mathbf{R}^1

12.2.7 *Some Examples to Illustrate the Definition of $\int_S f(x) dx$*

Example 1
Example 2
Example 3
Example 4

12.2.8 *Integrals over Regions in \mathbf{R}^2*

12.2.9 *Exercises on Double Integrals*

Exercise 1: Evaluating a Double Integral on a Triangle
Exercise 2: Evaluating a Double Integral on a Triangle
Exercise 3: Double Integral on a Circular Segment
Exercise 4: Double Integral on a Circular Sector
Exercise 5: Double Integral on a Half Ring
Exercise 6: Double Integral on a Triangle
Exercise 7: Double Integral on a Triangle
Exercise 8: Double Integral on a Triangle
Exercise 9: Inverting and then Evaluating a Double Integral
Exercise 10: Inverting and then Evaluating a Double Integral
Exercise 11: Inverting and then Evaluating a Double Integral
Exercise 12: Inverting a Double Integral
Exercise 13: Inverting a Double Integral

12.2.10 *Approximating Double Integrals by Sums*

Darboux's Theorem

Using A Double Integral to Find Area
 Revisiting Area of the Region Between two Graphs
 Using a Double Integral to Find the Value of a Metal Plate
 Using a Double Integral to Find Volume

12.2.11 Exercises on Applications of Double Integrals

Exercise 1: Finding an Area
 Exercise 2: Finding an Area
 Exercise 3: Finding an Area
 Exercise 4: Expressing a Volume in Terms of a Double Integral
 Exercise 5: The Plumber's Nightmare
 Exercise 6: Finding a Volume
 Exercise 7: Expressing a Volume in Terms of a Double Integral
 Exercise 8: Volume of the Standard 3-Simplex

Document: 12.3 The Gamma and Beta Functions

Movie:  **12.3 The Gamma and Beta Functions**

12.3.1 The Equation $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0$

The Equation $\lim_{x \rightarrow \infty} \frac{x^0}{e^x} = 0$

The Equation $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0$ When p Is Negative

The Equation $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0$ When p Is Positive

12.3.2 Introducing the Gamma Function

Definition of the Gamma Function
 Some Examples to Illustrate the Gamma Function
 A Harder Example
 The Convergence of the Integral $\int_0^{\infty} x^{a-1} e^{-x} dx$
 The Graph of the Gamma Function

12.3.3 Some Elementary Facts About the Gamma Function

The Recurrence Formula
 The Gamma Function and Factorials
 The Substitution $x = t^2$
 The Value of $\Gamma\left(\frac{1}{2}\right)$

12.3.4 Introducing the Beta Function

Definition of the Beta Function
 Some Examples to Illustrate the Beta Function
 The Convergence of the Integral $\int_0^1 t^{a-1} (1-t)^{b-1} dt$
 The Graph of the Beta Function

12.3.5 Some Elementary Facts About the Beta Function

Symmetry of the Beta Function
 The Substitution $u = ct$
 The Substitution $t = \sin^2 \theta$
 The Value of $B\left(\frac{1}{2}, \frac{1}{2}\right)$

12.3.6 The Relationship Between the Gamma and Beta Functions

Introducing the Relationship
 Proof of the Formula $\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$

12.3.7 Some Exercises on the Gamma and Beta Functions

Exercise 1: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Exercise 2: $\Gamma\left(\frac{13}{2}\right)$

Exercise 3: $\int_0^{\pi/2} \cos^8 \theta \sin^{12} \theta d\theta$

Exercise 4: $\int_0^{\pi/2} \cos^7 \theta \sin^{12} \theta d\theta$

Exercise 5: $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$

Exercise 6: $\int_0^1 \sqrt{1-x^4} dx$

Exercise 7: $\int_0^1 \frac{1}{\sqrt{1-x^4}} dx$

Exercise 8: $\int_0^{\infty} \frac{1}{\sqrt{1+x^4}} dx$

Exercise 9: $\iint_{Q^2} x^{p-1} y^{q-1} dx dy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)}$

Exercise 10: $\int_0^{\pi/2} \sin^p \theta d\theta = \int_{\pi/2}^{\pi} \sin^p \theta d\theta$

Exercise 11: $B(a, a) = \frac{1}{2^{2a-1}} B\left(a, \frac{1}{2}\right)$

Exercise 12: $\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a)\Gamma\left(a + \frac{1}{2}\right)$

Exercise 13: $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2} \pi$

Exercise 14: $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$

12.3.8 A Hard Fact About the Gamma Function

- Statement of the Hard Fact
- An Application of the Hard Fact

Document: 12.4 Changing Integrals to Polar Coordinates

Movie:  **12.4 Changing Integrals to Polar Coordinates**

12.4.1 Introducing the Change to Polar Coordinates

- A First Look Changing to Polar Coordinates
- A More Careful Description of the Regions of Integration
- Motivating the Formula for Changing to Polar Coordinates

12.4.2 Exercises on Polar Coordinates

- Exercise 1: Using Polars to Evaluate an Integral
- Exercise 2: Using Polars to Evaluate an Integral
- Exercise 3: Using Polars to Evaluate an Integral
- Exercise 4: Using Polars to Evaluate an Integral
- Exercise 5: Using Polars to Evaluate an Integral
- Exercise 6: Using Polars to Evaluate an Integral
- Exercise 7: Using Polars to Evaluate an Integral
- Exercise 8: Using Polars to Evaluate an Integral
- Exercise 9: Using Polars to Evaluate an Integral
- Exercise 10: Using Polars to Evaluate an Integral
- Exercise 11: Using Polars to Evaluate an Integral
- Exercise 12: Using Polars to Evaluate an Integral
- Exercise 13: Using Polars to Evaluate an Integral
- Exercise 14: Using Polars to Evaluate an Integral

Document: 12.5 Integration of a Function of Three Variables

Movie:  **12.5 Integration of a Function of Three Variables**

12.5.1 Iterated Integrals in Three Variables

- Iterated Integrals with Constant Limits
- More General Iterated Integrals

12.5.2 Some Examples of Iterated Integrals in Three Variables

Example 1: $\int_2^3 \int_0^1 \int_{-2}^1 (xy + 2yz) dy dx dz$

Example 2: $\int_0^{\pi/4} \int_0^{\pi/3} \int_0^{\pi/2} \cos(x + y + z) dx dy dz$

Example 3: $\int_0^1 \int_0^1 \int_0^1 \frac{1}{(x+y+z)^{5/2}} dz dx dy$

Example 4: $\int_0^2 \int_1^x \int_0^{\pi(2y)} x \cos(yz) dz dy dx$

Example 5: Some Meaningless Iterated Integrals

12.5.3 The Fichtenholz Theorem

12.5.4 Integration over Regions in \mathbf{R}^3

Definition of the Integral over a Region in \mathbf{R}^3

Darboux's Theorem

Using a Triple Integral to Find Volume

Using a Triple Integral to Find the Mass of a Region

Using a Triple Integral to Find the Value of a Metal Solid

12.5.5 Some Exercises on the Conversion of Triple Integrals to Iterated Integrals

Exercise 1: Setting up a Triple Integral

Exercise 2: A Return to the Plumber's Nightmare

Exercise 3: Setting up a Triple Integral

Exercise 4: Setting up a Triple Integral

Exercise 5: Integrating on the Standard 3-Simplex

12.5.6 Cylindrical Coordinates

Introduction to Cylindrical Coordinates

Cylindrical Coordinates with θ Changing

Cylindrical Coordinates with r Changing

Cylindrical Coordinates with z Changing

12.5.7 Exercises on Cylindrical Coordinates

Exercise 1: Using Cylindricals to Evaluate an Integral

Exercise 2: Using Cylindricals to Evaluate an Integral

Exercise 3: Using Cylindricals to Evaluate an Integral

12.5.8 Spherical Coordinates

Introduction to Spherical Coordinates

Spherical Coordinates with θ Changing

Spherical Coordinates with ρ Changing

Spherical Coordinates with φ Changing

12.5.9 Changing Integrals to Spherical Coordinates

A First Look at the Method

A More Careful Description of the Regions of Integration

Motivating the Formula for Changing to Spherical Coordinates

12.5.10 Exercises on Spherical Coordinates

Exercise 1: Using Sphericals to Evaluate an Integral

Exercise 2: Using Sphericals to Evaluate an Integral

Exercise 3: Using Sphericals to Evaluate an Integral

Exercise 4: Using Sphericals to Evaluate an Integral

Exercise 5: Using Sphericals to Evaluate an Integral

Exercise 6: Using Sphericals to Evaluate an Integral

Exercise 7: Using Sphericals to Evaluate an Integral

Exercise 8: Using Sphericals to Evaluate an Integral

Exercise 9: Using Sphericals to Evaluate an Integral


Exercise 10: Finding the Centroid of a Solid Region

Exercise 11: Finding the Moment of Inertia of a Solid Region

 **Document: 12.6 Changing Variable in a Multiple Integral**

Movie:  **12.6 Changing Variable in a Multiple Integral**

 **12.6.1 Introduction to the Change of Variable Formula**

 **12.6.2 The Change of Variable Theorem for Integrals of Functions of a Single Variable**

Introduction to the Change of Variable Formula

Review of the Change of Variable Formula for Integrals Between Limits

Some Notes About the Change of Variable Formula for Integrals Between Limits

The Function u May Be Increasing or Decreasing or Neither Increasing nor Decreasing

As x Runs from a to b , There Is No Reason to Expect that $u(x)$ Stays Between $u(a)$ and $u(b)$

The Quantity $u(x)$ Can Run Several Times Between $u(a)$ and $u(b)$

The Change of Variable Formula for Integration on Intervals

When the Function u Is Increasing

When the Function u Is Decreasing

Combining the Two Cases

What Happens if u is Neither Increasing nor Decreasing?

 **12.6.3 The Change of Variable Formula for Double Integrals**

Introduction to the Change of Variable Formula for Double Integrals

Revisiting the Change to Polar Coordinates to Illustrate the Change of Variable Formula

Motivating the Change of Variable Formula

 **12.6.4 Exercises on Change of Variable for Double Integrals**

Exercise 1: Integrating on a Parallelogram

Exercise 2: Integrating on an Elliptical Region

Exercise 3: Integrating on a Region Bounded by Parabolas and Hyperbolas

Exercise 4: Integrating on a Region Bounded by Straight Lines and Hyperbolas

Exercise 5: Integrating on the Standard 2-Simplex

Exercise 6: Converting an Integral on an Elliptical Region to an Integral on Q^2

 **12.6.5 The Change of Variable Formula for Triple Integrals**

Introduction to the Change of Variable Formula for Three Variables

Motivating the Change of Variable Formula

 **12.6.6 Exercises on Change of Variable for Triple Integrals**

Exercise 1: Applying the Change of Variable Formula to Sphericals

Exercise 2: Integrating on the Standard 3-Simplex

Exercise 3: Application to Dirichlet Integrals

 **Document: 12.7 Integrals on Parametric Regions**

Part 1 of the video includes the material up to the proof of Stokes theorem (Subsection 12.7.10).

Movie:  **12.7 Integrals on Parametric Regions Part 1**

Part 2 of the video includes the material from the examples on Stokes theorem (Subsection 12.7.11) till the end of the section.

Movie:  **12.7 Integrals on Parametric Regions Part 2**

 **12.7.1 Preliminary Statement**

 **12.7.2 A Quick Review of Curves and Surfaces**

A Quick Review of Parametric Curves

A Quick Review of Parametric Surfaces in R^2 or R^3

 **12.7.3 The Boundary of a Parametric Surface**

The Notation $[A, B]$ if A and B are Points in Space

The Boundary of the Standard 2-Simplex Q^2

The Boundary of a Rectangle in \mathbf{R}^2
The Boundary of a Parametric Surface in \mathbf{R}^2 or \mathbf{R}^3
When the Domain Region is Q^2
When the Domain Region is a Rectangle
A Formula for Integrating on the Boundary of a Surface

12.7.4 Some Examples of Boundaries of Parametric Surfaces

Example 1: The Unit Disk
Example 2: A Portion of a Paraboloid
Example 3: The Unit Sphere
Example 4: A Möbius Band

12.7.5 A Change of Variable Formula for Integrals on the Boundary of a Surface

Introduction to the Change of Variable Formula
Proving the Change of Variable Formula

12.7.6 Green's Theorem for Double Integrals

Simple Closed Curves and Jordan Regions
Positively Oriented Boundary of a Jordan Region
Three Examples of Positively Oriented Jordan Curves
Example 1: The Standard 2-Simplex Q^2
Example 2: A Rectangle
Example 3: The Unit Disk
Introduction to Green's Theorem
Green's Theorem on the Standard 2-Simplex Q^2
Green's Theorem on a Rectangle
Green's Theorem for Double Integrals

12.7.7 Some Exercises on Green's Theorem for Double Integrals

Exercise 1: Using Green's Theorem to Find Area
Exercise 2: Finding the Area of a Region
Exercise 3: Finding the Area of a Region
Exercise 4: Finding the Area of a Region
Exercise 5: Using Green's Theorem to Find a Centroid
Exercise 6: Finding the Centroid of a Region

12.7.8 Integrating on Parametric Surfaces

Introducing Integrals on Parametric Surfaces
Integrating on a Parametric Surface in \mathbf{R}^2
Example of an Integral on a Parametric Surface in \mathbf{R}^2
Integrating on a Parametric Surface in \mathbf{R}^2
Example of an Integral on a Parametric Surface in \mathbf{R}^3
Integrating a Vector Field on a Surface
Green's Theorem for Integrals on Parametric Surfaces

12.7.9 Green's Theorem for Integrals on Parametric Surfaces

12.7.10 Stokes' Theorem

Introduction to Stokes' Theorem
Statement of Stokes' Theorem
Proof of Stokes' Theorem

12.7.11 Some Examples to Illustrate Stokes' Theorem

Example 1: Stokes' Theorem on a Triangle
Example 2: Stokes' Theorem on a Portion of Paraboloid
Example 3: Stokes' Theorem on a Sphere
Example 4: Stokes' Theorem on a Möbius Band
Example 5: Stokes' Theorem on a Slipped Möbius Band

12.7.12 Solid Parametric Regions in \mathbf{R}^3

Definition of a Parametric Region in \mathbf{R}^3

— **12.7.13 Some Examples of Parametric Regions in \mathbf{R}^3**

Example 1
Example 2
Example 3
Example 4

— **12.7.14 Integrating on a Solid Parametric Region in \mathbf{R}^3**

Definition of the Integral of a Function on a Solid Parametric Region

— **12.7.15 The Boundary of a Solid Parametric Region in \mathbf{R}^3**

The Boundary of the Standard 3-Simplex Q^3
The Boundary of a Rectangular Box in \mathbf{R}^3
Defining The Boundary of a Solid Parametric Region in \mathbf{R}^3
The Boundary of the Unit Ball in \mathbf{R}^3

— **12.7.16 A Change of Variable Formula for Integrals on the Boundary of a Solid Parametric Region**

Introduction to the Change of Variable Theorem
A Needed Tool from Linear Algebra
Proving the Change of Variable Formula

— **12.7.17 The Gauss Divergence Theorem**

Introduction to the Gauss Divergence Theorem
The Divergence Theorem on the Standard 3-Simplex Q^3
The Divergence Theorem on a rectangular box
The Gauss Divergence Theorem for Parametric Regions
Proof of the Gauss Divergence Theorem for Parametric Regions
The Gauss Divergence Theorem for Triple Integrals
Proof of the Gauss Divergence Theorem for Triple Integrals

— **12.7.18 Examples to Illustrate the Gauss Divergence Theorem**

Example 1
Example 2
Example 3