Virtual Calculus Tutor

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Movie: 2.3 Properties of Limits

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Document: 2.4 Trigonometric Limits

Movie: 2.4 Trigonometric Limits

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Document: 2.5 Continuity

Movie: 2.5 Continuity

2.5.1 Introducing the Concept of Continuity

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Example 3: $f(t) = t+3$ for $t \neq 3$

Example 4: $f(t) = \frac{t^2 - 9}{t - 3}$ when $t - 3 \neq 0$

Example 5: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 6: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 2 & \text{if } t = 3 \end{cases}$

Example 7: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t < 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 8: $f(t) = \begin{cases} t + 3 & \text{if } t < 3 \\ 6 & \text{if } t = 3 \\ 2 - t & \text{if } t > 3 \end{cases}$

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- Statement of the Bolzano Intermediate Value Theorem
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- The Intermediate Value Property

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Exercise 4: $f(x) = |x^2 - 4|$ for $0 \leq x \leq 5/2$

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Overview of Chapter 3: Derivatives

Document: 3.1 Introduction to Derivatives

Movie: ***3.1 Introduction to Derivatives***

3.1.1 Definition of a Derivative

Motivating the Definition Using Slopes
Definition of the Derivative of a Function
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3.1.2 Some Examples of Derivatives

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Example 5: $f(x) = x^3$ for all x, find $f'(x)$
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3.1.3 The Power Rule

Introducing the Power Rule
The Power Rule for the Case $p = -5$
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3.1.5 The Leibniz Notation for Derivatives

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Introducing the Leibniz Notation for Derivatives
The Power Rule in Leibniz Notation
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Exercise 2: $\frac{d}{dx}\frac{5}{\sqrt[4]{x^7}}$

Exercise 3: $y = 8x^3 - 6x - 1$ tangent line problem

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Exercise 5: Tangent from $(-2,-21)$ to $y = x^2$

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Exercise 7: $f(x) = |x-3|$ no derivative at 3

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Exercise 9: $x^2 \sin\frac{1}{x}$ derivative at 0?

Document: 3.2 Elementary Facts About Derivatives

Movie: 3.2 Elementary Facts About Derivatives

3.2.1 The Rules for Differentiation

The Sum Rule
- Stating the Sum Rule
- Explaining the Sum Rule

The Difference Rule
- Stating the Difference Rule
- Explaining the Difference Rule

The Constant Multiple Rule
- Stating the Constant Multiple Rule
- Explaining the Constant Multiple Rule

The Product Rule
- Stating the Product Rule
- A Needed Fact About Limits
- Explaining the Product Rule

The Quotient Rule
- Stating the Quotient Rule
- Explaining the Quotient Rule
- An Optional Deeper Comment About the Proof of the Quotient Rule

3.2.2 Exercises on the Rules for Differentiation

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Exercise 4: Horizontal tangents to $y = \frac{x^2}{1+x^4}$

Exercise 5: $y = \frac{x}{x^2+4}$ tangent line problem

Exercise 6: $f(x) = (x-3)^2 g(x)$ tangent line problem

Exercise 7: $\frac{d}{dx} f(x)g(x)h(x)$ extended product rule

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3.2.3 Higher Order Derivatives

3.2.4 Exercises on Higher Order Derivatives

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Exercise 4: Expand $(1+x)^7$ using derivatives

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Document: 3.3 Derivatives of the Trigonometric Functions

Movie: 3.3 Derivatives of the Trigonometric Functions

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3.3.2 Derivatives of the Other Trigonometric Functions

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Finding the Derivative of $\cot$ Directly from the Definition

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Finding the Derivative of $\sec$ Directly from the Definition

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Finding the Derivative of $\csc$ Directly from the Definition

Summary of the Trigonometric Derivatives

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Exercise 2: $\frac{d}{dx} x^2 \sin x \cos x$

Exercise 3: $\frac{d}{dx}\frac{x \sin x}{1+x^2}$

Exercise 4: Horizontal tangents to $y = 2\cos^2 x + 2\cos x - 1$

Exercise 5: $\frac{d}{dx}\left((f(x)-\sin x)^2 + (g(x)-\cos x)^2\right)$

Document: 3.4 Derivative of a Composition

Movie: 3.4 Derivative of a Composition

3.4.1 Composition of Functions

3.4.2 Some Examples of Compositions

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Example 2: $f(x) = 1 + x^2$ for every number x and $g(u) = u^{100}$ for every number u

Example 3: $f(x) = 2^x$ for every number x and $g(u) = \log_2 u$ for $u > 0$

Example 4: $f(x) = \frac{x-2}{1-2x}$ whenever $x \neq \frac{1}{2}$ and $g(u) = \frac{u-3}{1-3u}$ for $u \neq \frac{1}{3}$

3.4.3 Statement of the Composition Rule

3.4.4 Some Examples to Illustrate the Composition Rule

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3.4.5 Motivating the Composition Rule

3.4.6 Using Leibniz Notation in the Composition Rule

3.4.7 A Return to the Earlier Examples on the Composition

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Example 3: $\frac{d}{dx}\sqrt{\sin x}$

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Document: 3.5 Inverse Functions

Movie: 3.5 Inverse Functions

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Document: 3.6 Derivatives of Exponential and Logarithmic Functions

Movie: 3.6 Derivatives of Exponential and Logarithmic Functions

3.6.1 The Key to the Differentiation of an Exponential Function

3.6.2 Approximate Differentiation an Exponential Function with a Computer Algebra System

Choosing a Computer Algebra System
Setting up Scientific Notebook
Approximate Evaluation of $\frac{d}{dx}2^x$
Approximate Evaluation of $\frac{d}{dx}3^x$

3.6.2 Approximate Differentiation an Exponential Function with a Computer Algebra System Interactive Form

Choosing a Computer Algebra System
Setting up Scientific Notebook
Approximate Evaluation of $\frac{d}{dx}2^x$
Approximate Evaluation of $\frac{d}{dx}3^x$

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Comparing the Graphs $y = a^x$ and $y = \frac{d}{dx}a^x$
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3.6.3 Adjusting the Base of an Exponential Function: The Number e Interactive Form

Preliminary Note
Our Objective: To Obtain $\frac{d}{dx}a^x = 1a^x$
Adjusting the Base Numerically
Adjusting the Base Geometrically: Animation Method
Adjusting the Base Geometrically: Zooming Method
Comparing the Graphs $y = a^x$ and $y = \frac{d}{dx}a^x$
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Exercise 6: $\frac{d}{dx}\log|\sec x|$
Exercise 7: $\frac{d}{dx}\log|\sec x + \tan x|$
Exercise 8: $\frac{d}{dx}\log|\csc x + \cot x|$
Exercise 9: $\frac{d}{dx}(1+x^2)^{\sin x}$
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Document: 3.7 Inverse Trigonometric Functions

Movie: 3.7 Inverse Trigonometric Functions

3.7.1 The Function arccos

3.7.2 Some Examples to Illustrate the Function arccos

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3.7.3 Some Properties of the Function arccos

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The Derivative of the Function arccos
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3.7.4 The Function arcsin

3.7.5 Some Examples to Illustrate the Function arcsin

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3.7.6 Some Properties of the Function arcsin

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The Graph of the Function arcsin

3.7.7 The Function arctan

3.7.8 Some Examples to Illustrate the Function arctan

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The Numbers $\arctan 1$ and $\arctan(-1)$

The Numbers $\arctan \sqrt{3}$ and $\arctan\left(-\sqrt{3}\right)$

The Numbers $\arctan \frac{1}{\sqrt{3}}$ and $\arctan\left(-\frac{1}{\sqrt{3}}\right)$

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3.7.9 Some Properties of the Function arctan

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3.7.10 The Function arcsec

3.7.11 Some Examples to Illustrate the Function arcsec

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Document: 3.8 Implicit Functions

Movie: 3.8 Implicit Functions

3.8.1 Implicit 2D Graphs

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Example 2: $x^2 y - y^2 + xy^3 = 5$
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Example 5: $x^5 + y^5 - 3x^2 y = 0$
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3.8.2 The Implicit Function Theorem

3.8.3 Some Exercises on Implicit Functions

Exercise 1: Tangent to $x^2 + y^2 = 25$ at $(3,4)$
Exercise 2: Slope of $x^2 y - y^2 + xy^3 = 5$ at a general point (x,y)
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Exercise 7: Slope of $x\sin(x^2+y^2) + y = 0$ at a general point (x,y)

Document: 3.9 Hyperbolic Functions

Movie: 3.9 Hyperbolic Functions

3.9.1 Introduction to Hyperbolic Functions

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The Definitions of the Hyperbolic Functions

3.9.2 Arithmetical Properties of the Hyperbolic Functions

Behaviour of the Hyperbolic Functions at 0
"Pythagorean Identities" for the Hyperbolic Functions
Replacing x by $-x$ in the Hyperbolic Functions
Hyperbolic Function Values at a Sum or Difference
Analogues for the Hyperbolic Functions of the Trigonometric Double and Triple Angle Identities

3.9.3 Derivatives of the Hyperbolic Functions

The Equation $\frac{d}{dx}\sinh x = \cosh x$
The Equation $\frac{d}{dx}\cosh x = \sinh x$
The Equation $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
The Equation $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$

3.9.4 Inverse Functions of the Hyperbolic Functions

The Function arcsinh
Finding $\frac{d}{dx}\operatorname{arcsinh} x$
The Function arccosh
Finding $\frac{d}{dx}\operatorname{arccosh} x$
The Function arctanh
Finding $\frac{d}{dx}\operatorname{arctanh} x$
The Function arcsech
Finding $\frac{d}{dx}\operatorname{arcsech} x$

3.9.5 Some Derivatives that Involve the Hyperbolic Functions

Example 1: $\frac{d}{dx}\arctan(\sinh x)$
Example 2: $\frac{d}{dx}\arctan(e^x)$
Example 3: $\frac{d}{dx}\arcsin(\operatorname{sech} x)$
Example 4: $\frac{d}{dx}\log(\cosh x)$
Example 5: $\frac{d}{dx}\log(\sinh x)$
Example 6: $\frac{d}{dx}\operatorname{arcsec}(\cosh x)$
Example 7: $\frac{d}{dx}\arccos(\operatorname{sech} x)$
Example 8: $\frac{d}{dx}\operatorname{arccosh}(\sec x)$
Example 9: $\frac{d}{dx}\operatorname{arctanh}(\sin x)$

Overview of Chapter 4: Applications of the Derivative

Document: 4.1 Monotone Functions

Movie: 4.1 Monotone Functions

4.1.1 The Graph of a Function with a Positive Derivative

The Positive Derivative Principle
Looking at the Positive Derivative Principle Intuitively
A Note of Caution

4.1.2 Increasing and Decreasing Functions

Strictly Increasing Functions
Increasing Functions
Strictly Decreasing Functions
Decreasing Functions
Monotone Functions

4.1.3 More General Version of the Positive Derivative Principle

4.1.4 Exercises on Monotone Functions

Exercise 1: $f(x) = x^2 - 4x - 5$ for all x
Exercise 2: $f(x) = x^3 - 3x^2$ for all x
Exercise 3: $f(x) = f(x) = |x^2 - 4x - 5|$ for all x
Exercise 4 $f(x) = |x^3 - 3x^2|$ for all x
Exercise 5: $f(x) = \left(\frac{\log x}{x}\right)^2$ for $x > 0$

4.1.5 An Application of the Positive Derivative Principle

The Inequality $e^x > 1$ when $x > 0$
The Inequality $e^x > 1 + x$ when $x > 0$
The Inequality $e^x > 1 + x + \frac{x^2}{2}$ when $x > 0$

The Inequality $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$ when $x > 0$
The General Case $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ when $x > 0$

4.1.6 Working out Some Important Limits

The Limit $\lim_{x\to\infty} \frac{e^x}{x}$
The Limit $\lim_{x\to\infty} \frac{e^x}{x^5}$
The Limit $\lim_{x\to\infty} \frac{e^x}{x^n}$
The Limit $\lim_{x\to\infty} \frac{\log x}{x}$
The Limit $\lim_{x\to\infty} \frac{(\log x)^{1000000}}{x}$
The Limit $\lim_{x\to 0+} x \log x$
The Limit $\lim_{x\to 0+} x(\log x)^{1000000}$

Document: 4.2 Drawing Graphs of Functions

Movie: 4.2 Drawing Graphs of Functions

4.2.1 Maxima and Minima

Definition of Maxima and Minima
Definition of Local Maxima and Minima

4.2.2 Fermat's Theorem

Statement of Fermat's Theorem
- Part 1: Positive derivative not at the right endpoint
- Part 2: Negative derivative not at the right endpoint
- Part 3: Positive derivative not at the left endpoint
- Part 4: Negative derivative not at the left endpoint
- Part 5: Conclusion

Using Fermat's Theorem
Critical Numbers of a Function

4.2.3 Some Examples to Illustrate Fermat's Theorem

Example 1: $f(x) = x^3$ for $-2 \le x \le 2$
Example 2: $f(x) = x^3$ for $-2 \le x < 2$
Example 3: $f(x) = \begin{cases} \frac{x-1}{2} & \text{if } 0 \le x \le 3 \\ 2x - 5 & \text{if } 3 \le x \le 4 \end{cases}$

4.2.4 Exercises on Graphs of Functions

Exercise 1: $f(x) = x^2 - 4x - 5$ for $-2 \le x \le 6$
Exercise 2: $f(x) = x^2 - 4x - 5$ for $3 \le x \le 6$
Exercise 3: $f(x) = |x^2 - 4x - 5|$ for $-2 \le x \le 6$
Exercise 4: $f(x) = x^3 - 3x^2$ for $-1 \le x \le 4$
Exercise 5: $f(x) = \frac{x^2}{1+x^2}$ for all x
Exercise 6: $f(x) = xe^{-x}$ for $x \ge -1$
Exercise 7: $f(x) = xe^{-x^2}$ for all x
Exercise 8: $f(x) = x^2e^{-x^2}$ for all x
Exercise 9: $f(x) = 3\sin^4 x - 2\sin^3 x$ for $0 \le x \le 2\pi$
Exercise 10: $f(x) = x(\log x)^2$ for $0 < x \le 2$
Exercise 11: $f(x) = x^{2/3}(6 - x)^{1/3}$ for $-1 \le x \le 7$

4.2.5 Concavity of Graphs

The Graph of a Function with a Positive Second Derivative
The Graph of a Function with a Negative Second Derivative
Points of Inflection

4.2.6 Exercises on Concavity

Exercise 1: $f(x) = x^3 - 3x^2$ for all x
Exercise 2: $f(x) = \frac{x^2}{1+x^2}$ for all x
Exercise 3: $f(x) = xe^{-x}$ for all x

Exercise 4: $f(x) = xe^{-x^2}$ for all x
Exercise 5: $f(x) = x^2e^{-x^2}$ for all x
Exercise 6: $f(x) = \log(1 + x^2)$ for all x
Exercise 7: $f(x) = (\log x)^2$ for $x > 0$
Exercise 8: $f(x) = x(\log x)^2$ for $x > 0$
Exercise 9: $f(x) = x(\log(x^2))^2 - 3x\log(x^2)$ for $x \neq 0$
Exercise 10: $f(x) = x^{2/3}(6 - x)^{1/3}$ for $-1 \leq x \leq 7$
Exercise 11: $f(x) = \frac{x \log x}{1+x^2}$ for $x > 0$
Exercise 12: Theoretical

Document: 4.3 Applied Maxima and Minima

Movie: 4.3 Applied Maxima and Minima

4.3.1 Elementary Exercises on Applied Maxima and Minima

Exercise 1: The Chicken Coop Problem
Exercise 2: The Box Problem
Exercise 3: The Cylindrical Can Problem
Exercise 4: The Rectangle in a Semicircle Problem
Exercise 5: The Isosceles Triangle in a Parabola Problem
Exercise 6: The Isosceles Triangle in a Circle Problem
Exercise 7: The Cone in a Hemisphere Problem
Exercise 8: The Triangle and Semicircle Problem
Exercise 9: The Road and Field Problem (Special Case)
Exercise 10: The Dimmer Switch Problem
Exercise 11: An Electric Circuit Problem

4.3.2 The General Road and Field Problem (and Deriving Snell's Law)

The Narrow Road Version of the Road and Field Problem
The Wide Road Version of the Road and Field Problem
The Road and Field Problem and the Laws of Refraction
Comparing the Wide Road Problem with the Narrow Road Problem

4.3.3 Making a Quadrilateral of Maximum Area

Maximizing the Area of a Quadrilateral with Given Sides
The Three Sticks Problem

4.3.4 The Ice Cream Problem: Maximum Minimum Problems About Cones

Background Information About Cones
Maximizing the Volume of a Cone with a Given Slant Height
Minimizing the Slant Height of a Cone with a Given Volume
Maximizing the Volume of a Cone with Given Surface Area
Filling the Cone with Ice Cream

4.3.5 Introducing The Soapbox Car Problem (See Section 8.4 for the full discussion.)

Document: 4.4 Antiderivatives (Indefinite Integrals)

Movie: 4.4 Antiderivatives (Indefinite Integrals)

4.4.1 Antiderivative of a Function

4.4.2 Some Examples of Antiderivatives

Example 1: Antiderivative with respect x of $6x$
Example 2: Another antiderivative with respect x of $6x$
Example 3: Antiderivative with respect x of $\cos x$
Example 4: Antiderivative with respect x of $\frac{1}{x}$ when $x > 0$
Example 5: Antiderivative with respect x of $\frac{1}{x}$ when $x < 0$
Example 6: Antiderivative with respect x of $\frac{1}{x}$ when $x \neq 0$
Example 7: Antiderivative with respect x of x^p when $p \neq -1$

4.4.3 The Key Fact About Antiderivatives

Statement of the Key Fact
Finding all Possible Antiderivatives of a Given Function

4.4.4 Some Examples of General Antiderivatives

Example 1: $\int x dx = \frac{x^2}{2} + c$

Example 2: $\int x^p dx = \frac{x^{p+1}}{p+1} + c$

Example 3: $\int \frac{1}{x} dx = \log|x| + c$

Example 4: $\int \cos x dx = \sin x + c$

Example 5: $\int \sin x dx = -\cos x + c$

Example 6: $\int \sec^2 x dx = \tan x + c$

Example 7: $\int \sec x \tan x dx = \sec x + c$

Example 8: $\int \tan x dx = \log|\sec x| + c$

Example 9: $\int \cot x dx = \log|\sin x| + c$

Example 10: $\int \sec x dx = \log|\sec x + \tan x| + c$

Example 11: $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

Example 12: $\int \frac{1}{1+x^2} dx = \arctan x + c$

Example 13: $\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + c$

4.4.5 Changing Variable to Find an Antiderivative

Motivating the Change of Variable Method: Example 1
Motivating the Change of Variable Method: Example 2
Motivating the Change of Variable Method: Example 3
Motivating the Change of Variable Method: Example 4
Motivating the Change of Variable Method: Example 5
Motivating the Change of Variable Method: Example 6
Motivating the Change of Variable Method: Example 7
Introducing the Change of Variable Method
Applying The Change of Variable Method

4.4.6 Some Exercises on Changing Variable

Exercise 1: $\int \sqrt{1+x^2}\, 2x dx$

Exercise 2: $\int \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$

Exercise 3: $\int \frac{x}{1+x^2} dx$

Exercise 4: $\int \cos^4 x \sin x dx$

Exercise 5: $\int \sqrt{\tan x}\, \sec^2 x dx$

Exercise 6: $\int \frac{\cos(\log x)}{x} dx$

Exercise 7: $\int e^x \sin(3e^x) dx$

Exercise 8: $\int (\log \sin x)^2 \cot x dx$

Exercise 9: $\int x\sqrt{x+3}\, dx$

Exercise 10: $\int \sqrt{\sin x}\, \cos x dx$

Exercise 11: $\int \sqrt{\sin x}\, \cos^3 x dx$

Exercise 12: $\int \sqrt{\sin x}\, \cos^5 x dx$

Exercise 13: $\int \sqrt{\cos x}\, \sin^5 x dx$

Exercise 14: $\int \sec^6 x \sqrt{\tan x}\, dx$

Exercise 15: $\int \sec^3 x \tan^5 x dx$

Exercise 16: $\int (1+x)dx$ (two ways)

Exercise 17: $\int \sin 2\theta d\theta$ (two ways)

Exercise 18: $\int \frac{1}{1-x^2}dx$

Exercise 19: $\int \sec x dx$

Exercise 20: $\int \csc x dx$

4.4.7 Antiderivatives that Involve Hyperbolic Functions

Exercise 1: $\int \cosh x dx = \sinh x + c$

Exercise 2: $\int \sinh x dx = \cosh x + c$

Exercise 3: $\int \operatorname{sech} x dx = 2\arctan(e^x) + c$

Exercise 4: $\int \tanh x dx = \log \cosh x + c$

Exercise 5: $\int \sqrt[3]{\tanh x}\, \operatorname{sech}^2 x dx$

Exercise 6: $\int \frac{1}{\sqrt{x^2+1}}dx = \operatorname{arcsinh} x + c$

Exercise 7: $\int \frac{\operatorname{arcsinh} x}{\sqrt{x^2+1}}dx$

Exercise 8: $\int \frac{\cos x}{\sqrt{1+\sin^2 x}}dx$

Exercise 9: $\int \frac{\sqrt{\operatorname{arccosh} x}}{\sqrt{x^2-1}}dx$

Exercise 10: $\int \frac{1}{1-x^2}dx = \operatorname{arctanh} x + c$

Exercise 11: $\int \frac{1}{x\sqrt{1-x^2}}dx = -\operatorname{arcsech} x + c$

Document: 4.5 Rates of Change

Movie: 4.5 Rates of Change

4.5.1 Interpreting the Derivative as a Rate of Change

4.5.2 Some Exercises on Derivatives as Rates of Change

Exercise 1. Inflating a Balloon: Part 1
Exercise 2. Inflating a Balloon: Part 2
Exercise 3. A Leaking Cone: Part 1
Exercise 4. A Leaking Cone: Part 2
Exercise 5. Water Evaporating from a Cone
Exercise 6. Growth of a Bacterial Colony
Exercise 7. Growth of Money in a Bank Account
Exercise 8. Radioactive Decay

Document: 4.6 Motion of a Particle in a Straight Line

Movie: 4.6 Motion of a Particle in a Straight Line

4.6.1 The Position Function of a Moving Particle

4.6.2 Examples to Illustrate Position Functions

Example 1: $f(t) = t^2$ for $-1 \le t \le 1$
Example 2: $f(t) = t^4$ for $-1 \le t \le 1$
Example 3: $f(t) = t^2$ for $t \ge 0$
Example 4: $f(t) = \sin t$ for $t \ge 0$

4.6.3 Velocity, Speed, and Acceleration of a Particle

4.6.4 Some Exercises on Velocity, Speed, and Acceleration

Exercise 1: $f(t) = t^2$ at each time t
Exercise 2: $f(t) = \sin t$ at each time t in the interval $[0, 6\pi]$
Exercise 3: $f'(t) = 5t$ for each time t
Exercise 4: $f''(t) = 20$ for every t

4.6.5 Expressing Velocity and Acceleration in Terms of Postion

An Example to Illustrate Velocity and Acceleration at a Point x
A Formula for Velocity in Terms of Position
Returning to the Example
A Formula for Acceleration in Terms of Position
Returning, Once Again, to the Example

4.6.6 Newton's Law

Introducing the Concept of Mass
Introducing the Concept of Force
Introduction to Newton's Law
 The Role of Force when Mass is Changing: The Sticky Ball Example
 The Role of Force when Velocity is Changing
Newton's Law of Motion when the Force Acts in the Direction of the Number Line
Newton's Law of Motion when the Force Acts Against the Direction of the Number Line
Units to Be Used in Newton's Law
 The Kilogram, the Newton, and the Meter
 The Gram, the Dyne, and the centimeter
 The Pound Mass, the Poundal, and the Foot
 The Slug, the Pound Force, and the Foot (Included Reluctantly)

4.6.7 Some Exercises on Newton's Law

Exercise 1: A Constant Mass Propelled by a Constant Force
Exercise 2: A Constant Mass Projected Upward Near the Ground
Exercise 3: A Sticky Ball Coasting in a Dust Cloud
Exercise 4: A Sticky Ball Coasting in a Resisting Dust Cloud
Exercise 5: Another Sticky Ball Problem
Exercise 6: A Particle Coasting in a Resisting Medium; Resistance Proportional to the Velocity
Exercise 7: A Particle Coasting in a Resisting Medium; Resistance Proportional to the Square of the Velocity
Exercise 8: A Rocket Problem
Exercise 9: A Particle Moving Away from the Earth
Exercise 10: A Relativistic Problem

Overview of Chapter 5: The Mean Value Theorem and its Applications

Document: 5.1 The Mean Value Theorem

Movie: 5.1 The Mean Value Theorem

5.1.1 Introduction to the Mean Value Theorem

Why Do We Need the Mean Value Theorem?
A Sneak Preview of the Mean Value Theorem
Statement of the Mean Value Theorem
The Speeding Ticket Problem

5.1.2 Rolle's Theorem

The Statement of Rolle's Theorem
Two Important Ingredients Needed for Rolle's theorem
 A Brief Restatement of Fermat's theorem
 A Brief Restatement of the Theorem on Maxima and Minima of Continuous Functions
Proof of Rolle's Theorem
A Two Function Version of Rolle's Theorem
Proof of the Mean Value Theorem

5.1.3 Proving the Positive Derivative Principle

Proof of Assertion 1
Proof of Assertion 2
Proof of Assertion 3
Proof of Assertion 4
Proof of Assertion 5

5.1.4 Some Exercises on the Mean Value Theorem

Exercise 1: A function with a maximum
Exercise 2: The derivative of a strictly increasing function
Exercise 3: Reversing the endpoints of the interval
Exercise 4: A condition for a function to be one-one
Exercise 5: Using the inequality $|f'(x)| \le 1$
Exercise 6: When the inequality $|f(t) - f(x)| \le |t - x|^2$ holds
Exercise 7: A condition for two functions to be sin and cos
Exercise 8: Derivatives have an intermediate value property
Exercise 9: A two function version of Exercise 8

Document: 5.2 Approximating a Function with Polynomials

Movie: 5.2 Approximating a Function with Polynomials

5.2.1 Introduction to Polynomials

Definition of a Polynomial
Expanding $(1+x)^8$: Motivating the Binomial Theorem
The Binomial Theorem

5.2.2 The Coefficients of a General Polynomial

Special Notation for Higher Derivatives of a Function
Finding the Coefficients of a Given Polynomial
The Degree of a Polynomial
Recentering the Terms of a Polynomial

5.2.3 Taylor Polynomials of a Function

Definition of The Taylor Polynomials

5.2.4 Some Examples of Taylor Polynomials

Example 1: $f(x) = 2 - 4x + 3x^2 + 7x^3 + 5x^4$ for each x
Example 2: $f(x) = \frac{1}{1+x^2}$ for each x
Example 3: $f(x) = \frac{1}{1+x^2}$, Taylor polynomials centered at 1
Example 4: Using a computer algebra system to find Taylor polynomials
Example 5: Another application of a computer algebra system

5.2.5 Finding The Remainder Term

Introducing the Remainder Term of a Taylor Polynomial
A Quick Review of Rolle's Theorem
A Version of Rolle's Theorem for the Second Derivative
A Version of Rolle's Theorem for the Third Derviative
A Version of Rolle's Theorem for the Fourth Derivative
Motivating the Higher Derivative Form of the Mean Value Theorem: A Mean Value Theorem for the Fourth Derivative
The Higher Deriverative Form of the Mean Value Theorem (Sometimes Called the Taylor Mean Value Theorem)

5.2.6 Some Applications of the Taylor Mean Value Theorem

Finding an Approximation to e
The Number e is Irrational
Finding an Approximation to e^3
Finding an approximation to $\log\left(\frac{3}{2}\right)$
Finding an approximation to $\log\left(\frac{1}{2}\right)$
Finding An Approximation to $\cos 1$
The Number $\cos 1$ Is Irrational

Document: 5.3 Indeterminate Forms

Movie: 5.3 Indeterminate Forms

5.3.1 Introduction to Indeterminate Forms

5.3.2 Some Examples to Illustrate Indeterminate Forms

Example 1: $\lim_{x\to 0} \frac{3x}{x} = 3$

Example 2: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Example 3: $\lim_{x\to 0+} x(\log x) = 0$

Example 4: $\lim_{x\to\infty} \frac{(\log x)^3}{x} = 0$

Example 5: $\lim_{x\to 0}(1+2x)^{1/x} = e^2$

Example 6: $\lim_{x\to\infty}\left(\sqrt{x^2+3x+1} - \sqrt{x^2-2x+7}\right) = \frac{5}{2}$

5.3.3 L'Hôpital's rule

Introducing L'Hôpital's rule
More Careful Statement of L'Hôpital's rule
Some Remarks About L'Hôpital's rule
- The Rule Works for One-Sided and Two-Sided Limits
- The Limit May Be Finite or Infinite
- The Case in Which $\lim_{x\to a} g(x) = \infty$
- A Brief History of L'Hôpital's rule

A Special Case of L'Hôpital's Rule
- Example 1 Showing Use of the Special Case of L'Hôpital's Rule
- Example 2 Showing Use of the Special Case of L'Hôpital's Rule
- Proof of the Special Case of L'Hôpital's Rule

5.3.4 Exercises on Indeterminate Forms

Exercise 1: $\lim_{x\to\infty} \frac{3x-7}{2x+5}$

Exercise 2: $\lim_{x\to 0} \frac{e^x-1}{x}$

Exercise 3: $\lim_{x\to 0} \frac{e^x \sin 5x - \sin 3x}{x}$

Exercise 4: $\lim_{x\to 0}\left(\frac{\tan x - x}{x - \sin x}\right)$

Exercise 5: $\lim_{x\to 0}\left(\frac{x - \sin x}{x^3}\right)$

Exercise 6: $\lim_{x\to 0+} x \log x$

Exercise 7: $\lim_{x\to 1} \frac{\log x}{x-1}$

Exercise 8: $\lim_{x\to\infty} \frac{\log x}{x}$

Exercise 9: $\lim_{x\to\infty} \frac{(\log x)^2}{x}$

Exercise 10: $\lim_{x\to\infty} \frac{(\log x)^p}{x} = 0$

Exercise 11: $\lim_{x\to\infty}\left(\log(3x+2) - \log(2x-5)\right)$

Exercise 12: $\lim_{x\to\infty} \frac{\log(x+2)}{\log(x-5)}$

Exercise 13: $\lim_{x\to\infty}\left(\frac{(\log(3x+2))^2 - (\log(2x-5))^2}{\log x}\right)$

Exercise 14: $\lim_{x\to\infty} \frac{\exp\left(\sqrt{\log x}\right)}{x}$

Exercise 15: $\lim_{x\to\pi/2}(\sin x)^{\tan x}$

Exercise 16: $\lim_{x\to\infty} x^{(\log x)/x}$

Exercise 17: $\lim_{x\to 0}\left((1+px)^{1/x}\right)$

Exercise 18: $\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{x}$

Exercise 19: $\lim_{x\to\infty}\left((\log(x+1))^2 - (\log x)^2\right)$

Exercise 20: $\lim_{x\to\infty} \frac{(x+1)^{\log(x+1)}}{x^{\log x}}$

Exercise 21: $\lim_{x\to\infty}\left(\frac{\log(x+1)}{\log x}\right)^x$

Exercise 22: $\lim_{x\to\infty} \frac{xe^{\sin x}}{\log x}$

***5.3.5 An Important Limit**: $\lim_{x\to\infty} x\left(1 - \frac{x^p}{(x+1)^p}\right)$*

Overview of Chapter 6: Integrals

Document: 6.1 Introducing Integrals as Antiderivatives

Movie: 6.1 Introducing Integrals as Antiderivatives

6.1.1 Preliminary Note: This Movie Takes the Fast Track into Integral Calculus

6.1.2 Defining Integrals Using Antiderivatives

Reviewing a Property of Antiderivatives
Defining the Symbol $\int_a^b f(x)dx$
Notation for Taking a Function Between Limits
The Symbol x is Not Important

6.1.3 Some Examples to Illustrate the Definition of an Integral

Example 1: $\int_2^5 xdx$

Example 2: $\int_0^{\pi/2} \cos xdx$

Example 3: $\int_2^9 \frac{1}{x}dx$

Exàmple 4: $\int_0^{\pi/4} \sec^2 xdx$

Example 5: $\int_0^{\pi/4} \sec x \tan xdx$

Example 6: $\int_0^{\pi/4} \sec xdx$

Example 7: $\int_{-1}^2 2x\sqrt{1+x^2}\,dx$

6.1.4 Linearity and Additivity of the Integral

Linearity of the Integral
Additivity of the Integral
The Symbol $\int_b^a$ when $a < b$

6.1.5 Using Integrals to Find Area

The Area Under the Graph of a Nonnegative Function: Historical Approach Using Infinitesimals
The Area Under the Graph of a Nonnegative Function Without Using Infinitesimals
Area of the Region Between Two Graphs
Area Between the Graph of a Negative Function and the x-Axis

6.1.6 Some Exercises on Area

Exercise 1: The region between $y = 4 - x^2$ and the x-axis
Exercise 2: A triangular region
Exercise 3: Region between $y = x^3 - 3x^2 + 2$ and $y = -x^2 + 3x + 2$
Exercise 4: Region between $y = \sin x$ and $y = \cos x$
Exercise 5: Region between $y = \sin x$ and $y = \sin 2x$
Exercise 6: Region between $y = \sin x$ and $y = \sqrt{\sin x}\cos x$

6.1.7 Derivatives of Integrals: The Equation $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

6.1.8 Exercises on Derivatives of Integrals

Exercise 1: $\frac{d}{dx}\int_1^x \sqrt{1+t+t^4}\,dt$

Exercise 2: $\frac{d}{dx}\int_1^x \sqrt{1+t+t^4}\,dt$

Exercise 3: $\frac{d}{dx}\int_2^{\sin x} \sqrt{1+t+t^4}\,dt$

Exercise 4: $\frac{d}{dx}\int_{2}^{\log x} \sqrt[3]{1+\sin^2 t}\, dt$

Exercise 5: $\frac{d}{dx}\int_{\exp(\sin x)}^{5} \sqrt[3]{1+t^2}\, dt$

Exercise 6: $\frac{d}{dx}\int_{\sin x}^{\exp\left(x^2\right)} \sqrt{1+t^4}\, dt$

Document: 6.2 Riemann Sums

Movie: 6.2 Riemann Sums

6.2.1 Summation Notation

Introducing Summation Notation
Some Simple Examples to Illustrate Summation Notation

Example 1: $\sum_{j=3}^{5} j^3$

Example 2: $\sum_{j=0}^{7} (-1)^j$

Example 3: $\sum_{j=1}^{n} 4$

Arithmetical Rules for Summation

Working Out the Sum $\sum_{j=1}^{n} j$

Another Way of Working Out $\sum_{j=1}^{n} j$

Working Out the Sum $\sum_{j=1}^{n} j^2$

Working Out the Sum $\sum_{j=1}^{n} j^3$

Using a Computer Algebra System to Work Out $\sum_{j=1}^{n} j^p$

6.2.2 Introduction to Riemann Sums

Motivating Riemann Sums
Definition of a Partition
Definition of a Riemann Sum
Regular Partitions
Darboux's Theorem
Left Sums, Right Sums, and Midpoint Sums
Left Sums
Right Sums
Midpoint Sums

6.2.3 Some Examples to Illustrate Darboux's Theorem

Example 1: $\int_0^1 x dx$

Example 2: $\int_0^1 x^2 dx$

Example 3: $\int_a^b x^2 dx$

Example 4: $\int_0^1 \sqrt{x}\, dx$

Example 5: $\int_0^1 \sqrt[3]{x^2}\, dx$

Document: 6.3 Riemann Sums with a Computer Algebra System

Movie: 6.3 Riemann Sums with a Computer Algebra System

6.3.1 Introductory Comment

6.3.2 Setting up The Riemann Sums

Supplying the Regular Partition to a Computer Algebra System
Introducing a Temporary Function f
Defining the Left Sum of a Function
Defining the Right Sum of a Function
Defining the Trapezoidal Sum of a Function
Defining Midpoint Sum of a Function
Defining the Simpson Sum of a Function
Motivation of the Simpson Sum

6.3.3 Numerical Approximations to Integrals

Summary of the Definitions
Using the Sums to Estimate $\int_0^1 \sqrt[3]{1+x^2}\,dx$
Using the Sums to Estimate $\int_0^1 \sqrt[3]{1-x^2}\,dx$
Obtaining Arrays of Approximating Sums Automatically

Document: 6.4 Using Riemann Sums to Define an Integral

Movie: 6.4 Using Riemann Sums to Define an Integral

6.4.1 Our Objective in this Section

6.4.2 A Quick Review of Riemann Sums

Bounded Functions
Definition of a Partition
Definition of a Riemann Sum
Regular Partitions

6.4.3 Squeezing a Function, Integrability, and the Integral

Motivating the Idea of Squeezing
Definition of a Squeezing Pair of Sequences
A Key Fact About a Squeezing Pair of Sequences
Integrability and the Integral

6.4.4 Some Examples to Illustrate Integrability

Example 1: The Integral $\int_0^1 x dx$
Example 2: The Integral $\int_0^1 x^2 dx$
Example 3: Increasing Functions Are Integrable
Example 4: Decreasing Functions Are Integrable
Example 5: Continuous Functions Are Integrable
Example 6: A Function that Fails to be Integrable

6.4.5 Some Facts About the Integral

Linearity of the Integral
Nonnegativity of the integral
Additivity of the Integral
Darboux's Theorem

6.4.6 The Fundamental Theorem of Calculus

Part 1 of the Fundamental Theorem of Calculus
Part 2 of the Fundamental Theorem of Calculus

6.4.7 Optional Item: Error Estimates for the Simpson Sum (Not included in the video)

Background for the Error Estimates
An Example to Illustrate the Error Estimates

Overview of Chapter 7: Evaluating Integrals

Document: 7.1 Evaluating Integrals by Substitution

Movie: 7.1 Evaluating Integrals by Substitution

7.1.1 Some Common Antiderivatives

The Antiderivative $\int x^p dx$ when $p \neq -1$
The Antiderivative $\int e^x dx$
The Antiderivative $\int x^p dx$ when $p = -1$
The Antiderivative $\int \cos x dx$
The Antiderivative $\int \sin x dx$
The Antiderivative $\int \sec^2 x dx$
The Antiderivative $\int \sec x \tan x dx$
The Antiderivative $\int \tan x dx$
The Antiderivative $\int \cot x dx$
The Antiderivative $\int \sec x dx$
The Antiderivative $\int \frac{1}{\sqrt{1-x^2}} dx$
The Antiderivative $\int \frac{1}{1+x^2} dx$
The Antiderivative $\int \frac{1}{x\sqrt{x^2-1}} dx$

The List of Antiderivatives

7.1.2 Changing Variable to Calculate an Integral

Introducing the Change of Variable Method
Applying the Change of Variable Method

7.1.3 Some Exercises on the Change of Variable Method

Exercise 1: $\int_0^{\pi/2} \sin^2 x \cos x dx$

Exercise 2: $\int_0^1 \sqrt{1+x^2}\, 2x dx$

Exercise 3: $\int_0^1 \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$

Exercise 4: $\int_1^2 \frac{x}{1+x^2} dx$

Exercise 5: $\int_0^{\pi} \cos^4 x \sin x dx$

Exercise 6: $\int_0^{\pi/4} \sqrt{\tan x}\, \sec^2 x dx$

Exercise 7: $\int_0^{\pi/3} \tan^2 x dx$

Exercise 8: $\int_0^{\pi/3} \tan^3 x dx$

Exercise 9: $\int_0^{\pi/4} \tan^4 x dx$

Exercise 10: $\int_1^{\exp(\pi/3)} \frac{\cos(\log x)}{x} dx$

Exercise 11: $\int_{\log(\pi/12)}^{\log(\pi/6)} e^x \sin(3e^x) dx$

Exercise 12: $\int_0^1 x\sqrt{x+3}\, dx$

Exercise 13: $\int_0^{\pi/2} \sqrt{\sin x}\, \cos x dx$

Exercise 14: $\int_0^{\pi/2} \sqrt{\sin x}\, \cos^3 x dx$

Exercise 15: $\int_0^{\pi/2} \sqrt{\sin x}\, \cos^5 x dx$

Exercise 16: $\int_0^{\pi/2} \sqrt{\cos x}\, \sin^5 x dx$

Exercise 17: $\int_0^{\pi/2} \cos^2 x dx$

Exercise 18: $\int_0^{\pi/2} \sin^4 x \cos^2 x dx$

Exercise 19: $\int_0^{\pi} \sqrt[3]{1+2\sin^2 x+\sin^5 x}\, \cos x dx$

Exercise 20: $\int_0^{\pi/4} \sec^6 x \sqrt{\tan x}\, dx$

Exercise 21: $\int_0^{\pi/3} \sec^3 x \tan^5 x dx$

Exercise 22: $\int_0^{\pi/3} \sec x dx$

Exercise 23: $\int_0^{1/2} \frac{\sqrt[3]{\arcsin x}}{\sqrt{1-x^2}} dx$

Exercise 24: $\int_1^{\sqrt{3}} \frac{1}{(1+x^2)\arctan x} dx$

Exercise 25: $\int_0^1 \frac{\arctan x}{(1+x^2)\sqrt{1+(\arctan x)^2}} dx$

Exercise 26: $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}\,\operatorname{arcsec} x} dx$

Document: 7.2 Evaluating Integrals by Parts

Movie: 7.2 Evaluating Integrals by Parts

7.2.1 Introduction to Integration by Parts

7.2.2 Some Examples to Illustrate Integration by Parts

Example 1: $\int_0^{\pi/2} x \cos x dx$

Example 2: $\int_0^1 x e^{3x} dx$

Example 3: $\int_0^{\pi/2} \cos^2 x dx$

7.2.3 Explaining Integration by Parts

Explaining Integration by Parts for Integrals
Explaining Integration by Parts for Antiderivatives

7.2.4 Exercises on Integration by Parts

Exercise 1

Exercise 1 Part a: $\int_0^{\pi/2} x^2 \cos x dx$

Exercise 1 Part b: $\int x^2 \cos x dx$

Exercise 2

Exercise 2 Part a: $\int_1^2 x \log x dx$

Exercise 2 Part b: $\int x \log x dx$

Exercise 3

Exercise 3 Part a: $\int_1^2 x(\log x)^2 dx$

Exercise 3 Part b: $\int x(\log x)^2 dx$

Exercise 4

Exercise 4 Part a: $\int_1^2 x(\log x)^3 dx$

Exercise 4 Part b: $\int x(\log x)^3 dx$

Exercise 5: $\int_1^2 \log x dx$

Exercise 6: $\int_0^{\pi^2/4} \cos\sqrt{x}\, dx$

Exercise 7: $\int_0^1 \arctan x dx$

Exercise 8: $\int_0^1 x \arctan x dx$

Exercise 9: $\int_0^{1/2} \arcsin x dx$

Exercise 10: $\int_0^{\pi/2} x \sin x \cos x dx$

Exercise 11: $\int_0^1 x \arcsin x dx$

Exercise 12: $\int_0^{\pi} e^x \cos x dx$

Exercise 13: $\int_0^{\pi/3} \sec^3 x dx$

Exercise 14: $\int_0^{\log\sqrt{3}} \operatorname{sech}^3 x dx$

Exercise 15: $\int_0^{2\pi} \cos mx \cos nx dx$

7.2.5 Reduction Formulas

Introduction to Reduction Formulas
Example 1: A Reduction Formula for the Integral $\int_1^2 x(\log x)^n dx$
Example 2: A Reduction Formula for the Antiderivative $\int x(\log x)^n dx$
Example 3: A Reduction Formula for the Integral $\int_0^{\pi/2} x^n \cos x dx$
Example 4: A Reduction Formula for the Antiderivative $\int x^n e^x dx$
Example 5: A Reduction Formula for the Antiderivative $\int \cos^n x dx$
Example 6: A Reduction Formula for the Integral $\int_0^{\pi/2} \cos^n x dx$
Example 7: A Reduction Formula for the Antiderivative $\int \sin^n x dx$
Example 8: A Reduction Formula for the Integral $\int_0^{\pi/2} \sin^n x dx$
Example 9: A Reduction Formula for the Antiderivative $\int \tan^n x dx$
Example 10: A Reduction Formula for the Antiderivative $\int \cot^n x dx$
Example 11: A Reduction Formula for the Antiderivative $\int \sec^n x dx$
Example 12: A Reduction Formula for the Integral $\int_0^{\pi/4} \sec^n x dx$

7.2.6 Wallis' Formula: $\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{\sqrt{n}\,(2n)!} = \sqrt{\pi}$

Introduction to Wallis' Formula
A Return to the Integral $\int_0^{\pi/2} \cos^n x dx$
Deriving Wallis' Formula

Document: 7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

Movie Option 1: *7.3 Evaluating Integrals Using Trigonometric Substitutions Only*

Movie Option 2:

7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

7.3.1 Preliminary Notes

Introduction to this Section
How Do I Know Whether to Use Trig or Hyperbolic Substitutions?
How Do I Know Whether a Given Integral Is of Type 1, 2, or 3?

7.3.2 Substitutions Involving sin *or* tanh

Introduction to the sin Substitution
An Example to Illustrate the sin Substitution
Introduction to the tanh Substitution
An Example to Illustrate the tanh Substitution
Integrals of Expressions Involving $\sqrt{a^2 - x^2}$

7.3.3 Substitutions Involving sec *or* cosh

Introduction to the sec Substitution
An Example to Illustrate the sec Substitution
Introduction to the cosh Substitution
An Example to Illustrate the cosh Substitution
Integrals of Expressions Involving $\sqrt{x^2 - a^2}$

7.3.4 Substitutions Involving tan *or* sinh

Introduction to the tan Substitution
An Example to Illustrate the tan Substitution

Introduction to the sinh Substitution
An Example to Illustrate the sinh Substitution
Integrals of Expressions Involving $\sqrt{a^2+x^2}$

7.3.5 Exercises on Trigonometric and Hyperbolic Substitutions

Exercise 1: $\int_0^{3/2} \sqrt{9-x^2}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 2: $\int_0^{3} \sqrt{9-x^2}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution: **Omitted**

Exercise 3: $\int_0^{5/2} \frac{1}{\sqrt{25-x^2}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 4: $\int_0^{3} \frac{1}{9+x^2}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 5: $\int_{\sqrt{2}}^{2} \frac{x^2}{\sqrt{x^2-1}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 6: $\int_{3\sqrt{2}}^{6} \frac{1}{(x^2-9)^{3/2}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 7: $\int_0^{1} \frac{x}{(1+x^2)^{3/2}}\,dx$

Exercise 8: $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 9: $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$

Exercise 10: $\int_{1/2}^{1/\sqrt{2}} \frac{1}{x\sqrt{1-x^2}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 11: $\int_0^{1} \frac{x^2}{(1+x^2)^{3/2}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 12: $\int_{3\sqrt{2}}^{6} \frac{1}{x\sqrt{x^2-9}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 13: $\int_{3\sqrt{2}}^{6} \frac{1}{x^2\sqrt{x^2-9}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 14: $\int_{3\sqrt{2}}^{6} \frac{1}{x^4\sqrt{x^2-9}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 15: $\int_{3\sqrt{2}}^{6} \frac{1}{\sqrt{x^2-9}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 16: $\int_{3\sqrt{2}}^{6} \frac{x}{\sqrt{x^2-9}}\,dx$

Exercise 17: $\int_{3\sqrt{2}}^{6} \frac{x^3}{\sqrt{x^2-9}}\,dx$

Exercise 18: $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 19: $\int_1^{2} \frac{\sqrt{x^2-1}}{x^4}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 20: $\int_5^{3+2\sqrt{3}} \frac{1}{\sqrt{x^2-6x+13}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 21: $\int_{1/2}^{2} \frac{1}{(2x^2-2x+5)^{3/2}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 22: $\int_{1+3\sqrt{2}}^{7} \frac{1}{\sqrt{x^2-2x-8}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 23: $\int_3^{5} \sqrt{6x-5-x^2}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution: **Omitted**

Exercise 24: $\int_1^{4/3} \frac{1}{(18x-9x^2-5)^{3/2}}\,dx$
Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Document: 7.4 Integration of Rational Functions

Movie: 7.4 Integration of Rational Functions

7.4.1 Background on Rational Functions

Introducing Rational Functions
Partial Fraction Expansions of Rational Functions

7.4.2 Some Exercises on Integration of Rational Functions

Exercise 1: $\int \frac{x+23}{x^2-3x-10}\,dx$

Exercise 2: $\int_0^1 \frac{3x^2+8x+7}{(x+1)(x+2)^2}\,dx$

Exercise 3: $\int_{-1}^1 \frac{x^2+x+2}{(x+3)(x^2+2x+5)}\,dx$

Exercise 4: $\int_{-1}^1 \frac{2x-2}{(x+3)(x^2+2x+5)}\,dx$

Exercise 5: $\int_{-1}^1 \frac{x^2+5x-2}{(x+3)(x^2+2x+5)}\,dx$

Exercise 6: $\int_0^{\pi/4} \sqrt{\tan x}\,dx$

Exercise 7: $\int_0^{\pi/4} \sqrt[3]{\tan x}\,dx$

7.4.3 Integrating Rational Functions of $\cos$ and $\sin$

7.4.4 Exercises on Rational Functions of $\cos\theta$ and $\sin\theta$

Exercise 1: $\int_0^{\pi/2} \frac{1}{\sin\theta+\cos\theta}\,d\theta$

Exercise 2: An Alternative Approach to $\int_0^{\pi/2} \frac{1}{\cos\theta+\sin\theta}\,d\theta$

Exercise 3: $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta$

Exercise 4: $\int_0^{\pi/2} \frac{\sin\theta}{1+\cos\theta+\sin\theta}\,d\theta$

Document: 7.5 Evaluating Improper Integrals

Movie: 7.5 Evaluating Improper Integrals

7.5.1 Introduction to Improper Integrals

Example 1: Motivating The integral $\int_0^1 \frac{1}{\sqrt{x}}dx$

Example 2: Motivating The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$

Example 3: Motivating The integral $\int_1^\infty \frac{1}{x^2}dx$

Definition of an Improper Integral

Definition of an Integral that Improper at its Right Endpoint

Definition of an Integral that Improper at its Left Endpoint

Convergence and Divergence of an Improper Integral

7.5.2 Some Examples of Improper Integrals

Example 1: $\int_0^1 \frac{1}{\sqrt{x}}dx$

Example 2: $\int_1^\infty \frac{1}{\sqrt{x}}dx$

Example 3: $\int_1^\infty \frac{1}{x^2}dx$

Example 4: $\int_0^\infty \cos x dx$

Example 5: $\int_0^1 \arcsin x dx$

7.5.3 Some Exercises on Improper Integrals

Exercise 1: $\int_0^\infty \frac{1}{(1+x^2)^{3/2}}dx$

The More Careful Approach

The Quick Approach

Exercise 2: $\int_2^\infty \frac{1}{x\sqrt{x^2-1}}dx$

The More Careful Approach

The Quick Approach

Exercise 3: $\int_1^2 \frac{1}{x\sqrt{x^2-1}}dx$

The More Careful Approach

The Quick Approach

Exercise 4: $\int_0^{\pi/2} \tan x dx$

The More Careful Approach

The Quick Approach

Exercise 5: $\int_0^{\pi/2} \sqrt{\tan x \sin x}\, dx$

The More Careful Approach

The Quick Approach

Exercise 6: $\int_0^{\pi/2} \frac{x\cos x - \sin x}{x^2}dx$

The More Careful Approach

The Quick Approach: **Omitted**

Exercise 7: $\int_0^\infty e^{-x}\sin x dx$

The More Careful Approach

The Quick Approach

Exercise 8: $\int_0^1 \frac{1}{x^p}dx$

Exercise 9: $\int_1^\infty \frac{1}{x^p}dx$

Exercise 10: $\int_2^\infty \frac{1}{x(\log x)^p}dx$

Exercise 11: $\int_0^2 \frac{1}{(x-1)^{1/3}}dx$

Exercise 12: $\int_0^1 \log x dx$

Document: 7.6 Convergence of Improper Integrals

Movie: 7.6 Convergence of Improper Integrals

7.6.1 Introduction to This Section

7.6.2 Convergence of Integrals of Nonnegative Functions

An Fundamental Principle About Integrals of Nonnegative Functions
Warning
An Example to Illustrate The Fundamental Principle
Introducing The Comparison Test for Improper Integrals
The Comparison Test for Improper Integrals
An Example to Illustrate the Comparison Test
A Second Example to Illustrate the Comparison Test
Introduction to the Limit Version of the Comparison Test
Statement of the Limit Comparison Test
Another Way of Looking at the Limit Comparison Test

7.6.3 Exercises on the Comparison Test

Exercise 1: $\int_1^\infty \frac{x}{x^3-3x^2+3x+7}dx$

Exercise 2: $\int_0^1 \frac{1}{\sqrt{x\cos x}}dx$

Exercise 3: $\int_1^\infty \frac{\log x}{x^2}dx$

Exercise 4: $\int_1^\infty \frac{1}{\sqrt[3]{x^2+5x+2}}dx$

Exercise 5: $\int_0^1 \frac{\sin^2 x}{x^{5/2}}dx$

Exercise 6: $\int_1^\infty \frac{\sqrt{x}}{x^2-x+1}dx$

Exercise 7: $\int_0^{\pi/2} \sqrt{\tan x}\,dx$

Exercise 8: $\int_1^2 \frac{1}{\log x}dx$

Exercise 9: $\int_0^{\pi/2} \log(\sin x)dx$

Exercise 10: $\int_1^\infty x^{\alpha-1}e^{-x}dx$

Exercise 11: $\int_0^1 x^{\alpha-1}e^{-x}dx$

Note On the Final Three Exercises of this Group

Exercise 12: $\int_2^\infty \frac{1}{(\log x)^{\log x}}dx$

Exercise 13: $\int_3^\infty \frac{1}{(\log\log x)^{\log x}}dx$

Exercise 14: $\int_3^\infty \frac{1}{(\log x)^{\log\log x}}dx$

7.6.4 Improper Integrals of Functions that Can Change Sign

Absolute Convergence of an Improper Integral
Every Absolutely Convergent Integral Must Converge
Conditional Convergence of an Improper Integral

7.6.5 Exercises on Absolute and Conditional Convergence of Improper Integrals

Exercise 1: $\int_1^\infty \frac{\sin x}{x^2}dx$ and $\int_1^\infty \frac{\cos x}{x^2}dx$

Exercise 2: $\int_1^\infty \frac{\sin x}{x}dx$

Exercise 10: The Polar Graph $r = \frac{1}{\theta}$
Exercise 11: The Polar Graph $r = \frac{1}{\sqrt{\theta}}$
Exercise 12: The Polar Graph $r = \cos 2\theta$
Exercise 13: The Polar Graph $r = \sin 3\theta$
Exercise 14: The Polar Graph $r = \cos 3\theta$
Exercise 15: The Polar Graph $r = 1 + \cos\theta$
Exercise 16: The Polar Graph $r = 1 + 2\cos\theta$
Exercise 17: A Computer Generated Polar Graph

8.3.9 Length of a Polar Graph

Introducing the Formula for Length of a Polar Graph
Example 1: Length of a Petal of the Graph $r = \cos 3\theta$.
Example 2: Length of a Cardioid
Example 3: Length of a Limacon

8.3.10 Area Bounded by a Polar Graph

8.3.11 Exercises on Area Bounded by a Polar Graph

Exercise 1: Area of a Petal of the Graph $r = \cos 3\theta$
Exercise 2: Area Enclosed by Cardioid
Exercise 3: Area Enclosed by a Spiral
Exercise 5: Area Enclosed by an Inward Spiral

Document: 8.4 The Soapbox Problem

Movie: 8.4 The Soapbox Problem

8.4.1 Introducing The Soapbox Car Problem

8.4.2 Preliminary Discussion: Maximizing a Special Kind of Rational Function

8.4.3 Finding the Kinetic Energy of a Rolling Wheel

The Nature of a Wheel in This Section
Kinetic Energy of a Stationary Spinning Wheel
The Kinetic Energy of a Rolling Wheel

8.4.4 The Dynamics of a Soapbox Car

Defining the Soapbox Car
The Equation of Motion of a Soapbox Car
Choosing the Radius to Maximize the Rolling Speed
A Final Note: Looking at The Extreme Cases

Document: 8.5 Conic Curves

Movie: 8.5 Conic Curves

8.5.1 Introduction to Conic Curves

8.5.2 Rectangular Equations of Conic Curves

A Rectangular Equation of a Parabola
A Rectangular Equation of an Ellipse
A Rectangular Equation of an Hyperbola
Asymptotes of an Hyperbola

8.5.3 Exercises on Conic Curves

Exercise 1: A Parametric Form of the Equation of an Ellipse
Exercise 2: Adding the Distances from a Point on an Ellipse to the Focal Points
Exercise 3: A Parametric Form of the Equation of an Hyperbola
Exercise 4: Parametric Form of an Hyperbola Using Hyperbolic Functions
Exercise 5: Subtracting the Distances from a Point on an Ellipse to the Focal Points
Exercise 6: The Reflection Property of a Parabola
Exercise 7: The Reflection Property of an Ellipse

8.5.4 Polar Equations of Conic Curves

The Case $\varepsilon = 0$
The Case $\varepsilon > 0$

Overview of Chapter 9: Sequences and Series

Document: 9.1 Limits of Sequences

Movie: ***9.1 Limits of Sequences***

9.1.1 Introducing the Concepts

Sequences and Sequence Notation
Introducing Limits of Sequences
Convergent Sequences and Divergent Sequences
Illustrating Convergent and Divergent Sequences

9.1.2 Elementary Facts About Limits of Sequences

Limit of a Constant Sequence
Relating Limits and Inequalities
The Sandwich Rule for Sequences
An Analogue of the Sandwich Rule for Infinite Limits
The Arithmetical Rules for Limits

9.1.3 Some Exercises on Limits of Sequences

Exercise 1: The Limit $\lim_{n\to\infty} \frac{(-1)^n}{n}$
Exercise 2: The Limit $\lim_{n\to\infty} (-1)^n$ Fails to Exist
Exercise 3: The Limit $\lim_{n\to\infty} \sqrt[n]{n}$
Exercise 4: The Limit $\lim_{n\to\infty} x^n$ when $x > 1$
Exercise 5: The Limit $\lim_{n\to\infty} x^n$ when $0 < x < 1$
Exercise 6: The Limit $\lim_{n\to\infty} x^n$ when $-1 < x < 1$
Exercise 7: The Limit $\lim_{n\to\infty} \frac{(-1)^n \log n}{n}$
Exercise 8: The Limit The Limit $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$
Exercise 9: The Limit $\lim_{n\to\infty} \frac{2^n}{n!}$
Exercise 10: The Limit $\lim_{n\to\infty} \frac{5^n}{n!}$
Exercise 11: The Limit $\lim_{n\to\infty} \frac{\log(n!)}{n^2}$
Exercise 12: The Important Limit $\lim_{n\to\infty} n\left(1 - \frac{n^p}{(n+1)^p}\right)$

9.1.4 Monotone Sequences

Introduction to Monotone Sequences
A Condition for an Increasing Sequence to Converge
A Final Note

Document: 9.2 An Intuitive Motivation of Infinite Series

Movie: ***9.2 An Intuitive Motivation of Infinite Series***

9.2.1 Our Objective in this Section

9.2.2 Some Examples to Illustrate Infinite Series

Example 1: The Sum $0 + 0 + 0 + 0 + 0 + 0 + \cdots$
Example 2: The Sum $1 + 1 + 1 + 1 + 1 + 1 + \cdots$
Example 3: Taking $a_n = \begin{cases} 1 & \text{if } 1 \le n \le 4 \\ 0 & \text{if } n \ge 5 \end{cases}$
Example 4: The Infinitely Repeating Decimal $0.\overline{1}$
Example 5: The Infinitely Repeating Decimal $0.\overline{9}$
Example 6: The Infinitely Repeating Decimal $0.\overline{473}$

Example 7: The Sum $1 + x + x^2 + x^3 + \cdots$ When $-1 < x < 1$
Example 8: The Sum $1 - x + x^2 - x^3 + x^4 - \cdots$ When $-1 < x < 1$
Example 9: The Sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$
Example 10: The sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
Example 11: The sum $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$
Example 12: The sum $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$
Example 13: The Equation $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
Example 14: The Equation $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
Example 15: The Equation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
Example 16: Comparing the Series Expansions of $\exp$, $\cos$, and $\sin$
Example 17: The Equation $x^2 = \frac{\pi^2}{3} - \frac{4\cos x}{1^2} + \frac{4\cos 2x}{2^2} - \frac{4\cos 3x}{3^2} + \cdots$

9.2.3 Concluding Remarks

Document: 9.3 Introduction to Infinite Series

Movie: 9.3 Introduction to Infinite Series

9.3.1 The Series with nth Term a_n

9.3.2 Convergence and Divergence of Series

9.3.3 Some Examples to Illustrate the Idea of a Series

Example 1: The Series $\sum 0$
Example 2: The Series $\sum 1$
Example 3: Taking $a_n = \begin{cases} 1 & \text{if } 1 \le n \le 4 \\ 0 & \text{if } n \ge 5 \end{cases}$
Example 4: The Series $\sum \frac{1}{n(n+1)}$
Example 5: The Series $\sum \frac{2}{n(n+1)(n+2)}$ for Each n
Example 6: The Geometric Series $\sum x^{n-1}$
Example 7: The Series $\sum \log\left(1 + \frac{1}{n}\right)$
Example 8: The Series $\sum (-1)^{n-1}$

9.3.4 The nth Term Criterion for Divergence

Introduction to the nth Term Criterion for Divergence
Proof of the nth Term Criterion for Divergence

9.3.5 A Return to the Examples of 9.3.3

Example 1: The Series $\sum 0$
Example 2: The Series $\sum 1$
Example 3: Taking $a_n = \begin{cases} 1 & \text{if } 1 \le n \le 4 \\ 0 & \text{if } n \ge 5 \end{cases}$
Example 4: The Series $\sum \frac{1}{n(n+1)}$
Example 5: The Series $\sum \frac{2}{n(n+1)(n+2)}$
Example 6: The Geometric Series $\sum x^{n-1}$
Example 7: The Series $\sum \log\left(1 + \frac{1}{n}\right)$
Example 8: The Series $\sum (-1)^{n-1}$

9.3.6 Some Applications of the nth Term Criterion for Divergence

A Ratio Criterion for Divergence
Testing the Series $\sum \frac{n!}{6^n}$
Divergence of the Series $\sum \frac{(2n)!}{(n!)^2}$
Divergence of The Series $\sum \frac{(-1)^n 4^n (n!)^2}{(2n)!}$
A Problem that We Cannot Solve Right Now: Test the Series $\sum \frac{(2n)!}{4^n (n!)^2}$
A Limit Form of the Ratio Criterion for Divergence
Divergence of the Series $\sum \frac{3^n}{n^{10}}$
Divergence of the Series $\sum \frac{(3^n)(n!)}{n^n}$

9.3.7 A Quick Summary of What We Know at Present

Document: 9.4 Convergence of Nonnegative Series

Movie: 9.4 Convergence of Nonnegative Series

9.4.1 Introduction to Nonnegative Series

9.4.2 The Integral Comparison Test

Divergence of the Series $\sum \frac{1}{n}$
Convergence of the Series $\sum \frac{1}{n^2}$
The General Form of the Integral Comparison Test
The p-Series
- The p-Series When $p > 1$
- The p-Series When $p < 1$
- Conclusion: Convergence Criteria for the p-Series

A Sharper Form of the p-Series
- The Case $p = 1$
- The Case $p < 1$
- The Case $p > 1$

9.4.3 Optional: A Sharper Type of Integral Comparison

An Extension of the Integral Comparison Test
Euler's Constant
The Limit $\lim_{n\to\infty} \sum_{j=n+1}^{2n} \frac{1}{j}$
Summing the Series $\sum \frac{(-1)^{n-1}}{n}$
Summing the Series $\sum \frac{1}{n(2n-1)}$

9.4.4 Comparing Series with One Another

The Comparison Test: Inequality Form
The Comparison Test: Limit Form

9.4.5 Some Exercises on The Comparison Test

Exercise 1: Testing the Series $\sum \frac{\sin^2 n}{n^2}$
Exercise 2: An Unsuccessful Attempt to Test the Series $\sum \frac{\sin^2 n}{n}$
Exercise 3: Testing the Series $\sum \frac{n}{n^4 + 7}$
Exercise 4: Testing the Series $\sum \frac{n}{n^4 - 7}$

Exercise 5: Testing the Series $\sum \frac{1}{n^{3/2}+n}$

Exercise 6: Testing the Series $\sum \frac{1}{n^{3/2}-n}$

Exercise 7: Testing the Series $\sum \frac{n}{\sqrt{n^4-n^2+2}}$

Exercise 8: Testing the Series $\sum \frac{\log n}{n^2}$

Exercise 9: Testing the Series $\sum \frac{n \log n}{\sqrt{n^5-n^2+2}}$

Exercise 10: Testing the Series $\sum \frac{1}{n^{1+1/n}}$

Exercise 11: Testing the Series $\sum \frac{1}{n^{1+(\log n)/n}}$

Exercise 12: Testing the Series $\sum \frac{1}{n^{1+(\log n)^2/n}}$

Exercise 13: Testing the Series $\sum \left(\frac{n}{n+1}\right)^n$

Exercise 14: Testing the Series $\sum \left(\frac{1}{\log n}\right)^3$

Exercise 15: Testing the Series $\sum \left(\frac{1}{\log n}\right)^n$

Exercise 16: Testing the Series $\sum \left(\frac{1}{\log n}\right)^{\log n}$

Exercise 17: Testing the Series $\sum \left(\frac{1}{\log \log n}\right)^{\log n}$

Exercise 18: Testing the Series $\sum \left(\frac{1}{\log n}\right)^{\log \log n}$

9.4.6 The Elementary Ratio Tests

Introducing the Ratio Tests
The Ratio Comparison Test
The d'Alembert Ratio Test, Inequality Form
The d'Alembert Ratio Test, Limit Form, Often Known as "The Ratio Test"

9.4.7 Some Exercises that Rely on d'Alembert's Test (Exercises on "The Ratio Test")

Exercise 1: Testing the Series $\sum \frac{n^{1000000}}{2^n}$

Exercise 2: Testing the Series $\sum \frac{2^n}{n!}$

Exercise 3: Testing the Series $\sum \frac{n!}{n^n}$

Exercise 4: Testing the Series $\sum \frac{n^{cn}}{n!}$ Given $c < 1$

Exercise 5: Testing the Series $\sum \frac{(2^n)(n!)}{n^n}$

Exercise 6: Testing the Series $\sum \frac{(3^n)(n!)}{n^n}$

Exercise 7: An Unsuccessful Attempt to Test the Series $\sum \frac{(e^n)(n!)}{n^n}$

Exercise 8: An Unsuccessful Attempt to Test the Series $\sum \frac{n^n}{(e^n)(n!)}$

Exercise 9: Testing the Series $\sum \frac{(2n)!}{5^n (n!)^2}$

Exercise 10: Testing the Series $\sum \frac{(2n)!}{3^n (n!)^2}$

Exercise 11: Testing the Series $\sum \frac{4^n (n!)^2}{(2n)!}$

Exercise 12: An Unsuccessful Attempt to Test the series $\sum \frac{(2n)!}{4^n (n!)^2}$

Exercise 13: Testing the Series $\sum \frac{((2n)!)^3}{((3n)!)^2}$

Exercise 14: Testing the Series $\sum \frac{(\log n)^n}{c^n (\log 2)(\log 3)\cdots(\log n)}$ for $c > 0$

Exercise 15: A Second Visit to the Series $\sum \frac{(e^n)(n!)}{n^n}$

9.4.8 The More Powerful Ratio Tests

Introduction to the More Powerful Tests
The Inequality Form of Raabe's Ratio Test
The Limit Form of Raabe's Test
A Level Two Ratio Test
A Level Three Ratio Test

9.4.9 Some Exercises on the More Powerful Ratio Tests

Exercise 1: Successful Testing of the Series $\sum \frac{(2n)!}{4^n (n!)^2}$

Exercise 2: Testing the Series $\sum \frac{|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)|}{n!}$

Exercise 3: Successful Testing of the Series $\sum \frac{n^n}{(e^n)(n!)}$

Exercise 4: Testing the Series $\sum \left(\frac{(2n)!}{4^n (n!)^2}\right)^2$

Document: 9.5 Absolute and Conditional Convergence

Movie: 9.5 Absolute and Conditional Convergence

9.5.1 Introduction to Convergence of Series Whose Terms Can Change Sign

9.5.2 Absolutely Convergent Series

Definition of an Absolutely Convergent Series
Convergence of Absolutely Convergent Series
Some Examples of Absolutely Convergent Series

9.5.3 Conditionally Convergent Series

9.5.4 The Alternating Series Test

Statement of the Alternating Series Test
Warning: Read the Statement of the Alternating Series Test Carefully!
Some Examples of Series Whose Conditional Convergence Can be Deduced from the Alternating Series Test
Proof of the Alternating Series Test
An Error Estimate for Alternating Series
Approximations to $\log 2$

9.5.5 Dirichlet's Test (Optional)

Statement of Dirichlet's Test
Proof of Dirichlet's Test
An Error Estimate for a Series Tested by Dirichlet's Test

9.5.6 Some Exercises on Dirichlet's Test (Optional)

Exercise 1: Testing the Series $\sum \frac{\sin nx}{n}$

Exercise 2: Testing the Series $\sum \frac{\cos nx}{n}$

Exercise 3: Testing the Series $\sum \frac{\cos^2 nx}{n}$

Exercise 4: Testing the Series $\sum \frac{\sin^2 nx}{n}$

Exercise 5: Conditional Convergence of $\sum \frac{\sin nx}{n}$ and $\sum \frac{\cos nx}{n}$

Exercise 6: A Relationship Between $\sum a_n$ and $\sum a_n^3$

9.5.7 Some Further Series to Test with the Alternating Series Test or Dirichlet's Test (Optional)

Introducing This Topic
A Special Technique for Testing Alternating Series
Testing the Series $\sum \frac{(-1)^n (2n)!}{4^n (n!)^2}$

Testing the Series $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$

Testing the Series $\sum \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right)(-1)^n$

Testing the Series $\sum \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)(-1)^n$

Document: 9.6 Power Series

Movie: 9.6 Power Series

9.6.1 Introduction to Power Series

9.6.2 Some Examples of Power Series

Example 1: The Geometric Series $\sum x^n$

Example 2: The Series $\sum \frac{x^n}{n!}$

Example 3: The Series $\sum \frac{(-1)^{n-1}x^n}{n}$

Example 4: The Series $\sum \frac{(-1)^{n-1}(x-4)^{2n-1}}{2n-1}$

Example 5: The Series $\sum n(x+3)^{n-1}$

Example 6: The Series $\sum (n!)x^n$

9.6.3 Radius and Interval of Convergence of a Power Series

The Case $0 < r < \infty$

The Case $r = 0$

The Case $r = \infty$

9.6.4 Some Exercises on Radius and Interval of Convergence

Exercise 1: The Series $\sum \frac{(x-5)^n}{2^n n^2}$

Exercise 2: The Series $\sum \frac{(x-5)^n}{2^n n}$

Exercise 3: The Series $\sum \frac{(-1)^n(x-5)^n}{2^n n}$

Exercise 4: The Series $\sum \frac{n(x-5)^n}{2^n}$

Exercise 5: The Series $\sum \frac{(x-5)^{2n}}{n3^n}$

Exercise 6: The Series $\sum \frac{(n!)^2}{(2n)!}x^n$

Exercise 7: The Series $\sum \frac{(2n)!}{(n!)^2}x^n$

Exercise 8: The Binomial Series $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n$

9.6.5 The Principal Facts About Power Series

The Derivative of the Sum of a Power Series

Higher Derivatives of the Sum of a Power Series

A Formula for the Coefficients of a Power Series

The Taylor and Maclaurin Series of a Given Function

9.6.6 Some Important Examples of Taylor Series

Example 1: The Geometric Series

Example 2: The Alternating Geometric Series

Example 3: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^{n+1} = \log(1+x)$, Found by the Derivative Method

Example 4: The Equation $\sum_{n=0}^{\infty} \frac{(1)^n}{2n+1}x^{2n+1} = \arctan x$, Found by the Derivative Method

Example 5: The Equation $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, Found by the Derivative Method

Example 6: The Equation $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, Found by the Remainder Method

Example 7: The Equations $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ Found by the Derivative Method

Example 8: The Equation $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ Found by the Remainder Method

Example 9: The Equation $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ Found by the Remainder Method

Example 10: A Bump Function

9.6.7 The Binomial Expansion

An Introduction to the Binomial Series
The Binomial Coefficients
A Needed Fact About the Binomial Coefficients
A Needed Fact About the Sum of the Binomial Series
Summing the Binomial Series

9.6.8 Abel's Theorem

9.6.9 Some Applications of Abel's Theorem

Example 1: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \log 2$

Example 2: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

Example 3: The Equation $\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} = \frac{1}{\sqrt{2}}$

9.6.10 Tauber's Theorem

Overview of Chapter 10: Some Basics in Linear Algebra

Document: 10.1 A Glance at Second and Third Order Determinants

Movie: ***10.1 A Quick Look at Second and Third Order Determinants***

10.1.1 Second Order Determinants

Definition of a Second Order Determinant
Example 1
Example 2
Solving Two Equations for Two Unknowns

10.1.2 Third Order Determinants

Definition of a Third Order Determinant
Alternative Expansions of a Third Order Determinant
Example of a Third Order Determinant
Solving Three Equations for Three Unknowns

10.1.3 More General Determinants

Document: 10.2 Vectors in Space

Movie: ***10.2 Vectors in Space***

Movie: 11.4 Partial Derivatives

11.4.1 Introduction to Partial Derivatives

Partial Derivatives of a Function of Two Variables
Functions of More than Two Variables
A Geometric Interpretation of Partial Derivatives
A More Precise Approach to Partial Derivatives
Higher Order Partial Derivatives
Equality of Second Order Mixed Partial Derivatives

11.4.2 Some Exercises on Partial Derivatives

Exercise 1: Working out Partial Derivatives
Exercise 2: Obtaining a Relationship among Partial Derivatives
Exercise 3: Obtaining a Relationship among Partial Derivatives
Exercise 4: Obtaining a Relationship among Partial Derivatives
Exercise 5: Obtaining a Relationship among Partial Derivatives
Exercise 6: Obtaining the Laplace Equation
Exercise 7: Obtaining the Laplace Equation
Exercise 8: The Cauchy-Riemann and Laplace Equations
Exercise 9: Failure of Equality of Mixed Second Order Partial Derivatives

11.4.3 The Chain Rule

An Example to Motivate the Chain Rule
A Second Example to Motivate the Chain Rule
The Chain Rule for Functions of Two Variables
The Chain Rule for Functions of Three Variables
The Chain Rule for Functions of n Variables

11.4.4 Some Exercises on the Chain Rule

Exercise 1: Illustrating the Chain Rule
Exercise 2: Illustrating the Chain Rule
Exercise 3: Changing to Polars
Exercise 4: A Linear Transformation
Exercise 5: Applying the Chain Rule to the Second Derivative
Exercise 6: Changing to Polars, Second Derivatives
Exercise 7: Changing to Sphericals, Second Derivatives
Exercise 8: Euler’s Formula for Homogeneous Functions

Document: 11.5 Vector Fields

Movie: 11.5 Vector Fields

11.5.1 Introduction to Vector Fields

The Force of Gravity as a Vector Field
Velocity of a Flowing Fluid as a Vector Field
Definition of a Vector Field
Scalar Fields

11.5.2 Some Examples of Vector Fields

Example 1: Plotting a Vector Field
Example 2: Plotting a Vector Field
Example 3: Plotting a Vector Field
Example 4: Plotting a Vector Field

11.5.3 Gradient, Divergence, Laplacian, and Curl

Gradient of a Real Function
Gradient of a Real Function
The Laplacian
The Curl of a Vector Field

An Application of the Preceding Maximum Minimum Problem

Overview of Chapter 12: Multivariable Integral Calculus

Document: 12.1 Integration on Curves

Movie: 12.1 Integration on Curves

12.1.1 Integration on a Smooth Curve

Definition of a Smooth Curve
Integrals of the Type $\int_P f dx$, $\int_P f dy$, and $\int_P f dz$
Integrals of the Type $\int_P F \cdot dP = \int_P F \cdot (dx, dy, dz) = \int_P f dx + g dy + h dz$
Application to Work Done by a Force

12.1.2 Examples of Integrals on Smooth Curves

Example 1
Example 2
Example 3

12.1.3 Fundamental Theorem of Calculus for Integrals on Curves

Introduction to the Fundamental Theorem
Statement of the Fundamental Theorem for Integrals of the Type $\int_P F \cdot dP$
Path Independence and the Fundamental Theorem
The Role of “Whirlpools”

12.1.4 Exercises on Integrals on Curves

Exercise 1: Evaluating an Integral on a Curve
Exercise 2: Integral on a Straight Line Segment
Exercise 3: Integrating a Conservative Field on an Unknown Curve
Exercise 4: Integrating a Conservative Field on an Unknown Curve
Exercise 5: Integrating a Non Conservative Field
Exercise 6: The Potential of the Force of Gravity

12.1.5 Reparametrizing a Curve

Motivating the Idea of a Reparametrization of a Curve
Reparametrizing a Curve in the Direction of Travel
Reparametrizing a Curve Reversing the Direction of Travel
An Animation to Illustrate a Reparametrization that Reverses the Direction of Travel
Integrating on a Reparametrization that is in the Direction of Travel
Integrating on a Reparametrization that Reverses the Direction of Travel

12.1.6 Integration on a Chain of Smooth Curves

Motivating the Idea of a Chain of Curves
Definition of a Chain of Curves
Integrating on a Chain of Curves
Integrating around a Triangle

12.1.7 Exercises on Integrals on Chains

Exercise 1: Evaluating an Integral on a Chain
Exercise 2: Integrating Around a Square
Exercise 3: An Integral Around a Triangle

12.1.8 A More General Notion of a Chain of Curves

Document: 12.2 Integration of a Function of Two Variables

Movie: 12.2 Integration of a Function of Two Variables

12.2.1 Iterated Integrals in Two Variables

Iterated Integrals with Constant Limits
More General Iterated Integrals

12.2.2 Some Examples of Iterated Integrals

Example 1: Evaluating an Iterated Integral
Example 2: Evaluating an Iterated Integral
Example 3: Evaluating an Iterated Integral
Example 4: Evaluating an Iterated Integral
Example 5: Evaluating an Iterated Integral
Example 6: Evaluating an Iterated Integral
Example 7: Evaluating an Iterated Integral
Example 8: Some Meaningless Iterated Integrals

12.2.3 The Fichtenholz Theorem

Note to Instructors on the Fichtenholz Theorem
Introduction to Fichenholz Theorem
Statement of the Fichtenholz Theorem

12.2.4 Some Exercises on Iterated Integrals

Exercise 1: Inverting the Order of an Iterated Integral
Exercise 2: Inverting the Order of an Iterated Integral
Exercise 3: Inverting the Order of an Iterated Integral
Exercise 4: Inverting the Order of an Iterated Integral
Exercise 5: Inverting the Order of an Iterated Integral
Exercise 6: Evaluating the Integral $\int_0^\infty e^{-x^2}dx$
Exercise 7: Failure of Equality of Iterated Integrals
Exercise 8: Failure of Equality of Iterated Integrals

12.2.5 Introduction to Integration over Regions

12.2.6 Integrals over Regions in $\boldsymbol{R}^1$

Integral over an Interval [a,b] in $\boldsymbol{R}^1$
The General Case of a Region in $\boldsymbol{R}^1$

12.2.7 Some Examples to Illustrate the Definition of $\int_S f(x)dx$

Example 1
Example 2
Example 3
Example 4

12.2.8 Integrals over Regions in $\boldsymbol{R}^2$

12.2.9 Exercises on Double Integrals

Exercise 1: Evaluating a Double Integral on a Triangle
Exercise 2: Evaluating a Double Integral on a Triangle
Exercise 3: Double Integral on a Circular Segment
Exercise 4: Double Integral on a Circular Sector
Exercise 5: Double Integral on a Half Ring
Exercise 6: Double Integral on a Triangle
Exercise 7: Double Integral on a Triangle
Exercise 8: Double Integral on a Triangle
Exercise 9: Inverting and then Evaluating a Double Integral
Exercise 10: Inverting and then Evaluating a Double Integral
Exercise 11: Inverting and then Evaluating a Double Integral
Exercise 12: Inverting a Double Integral
Exercise 13: Inverting a Double Integral

12.2.10 Approximating Double Integrals by Sums

Darboux's Theorem

Using A Double Integral to Find Area
Revisiting Area of the Region Between two Graphs
Using a Double Integral to Find the Value of a Metal Plate
Using a Double Integral to Find Volume

12.2.11 Exercises on Applications of Double Integrals

Exercise 1: Finding an Area
Exercise 2: Finding an Area
Exercise 3: Finding an Area
Exercise 4: Expressing a Volume in Terms of a Double Integral
Exercise 5: The Plumber's Nightmare
Exercise 6: Finding a Volume
Exercise 7: Expressing a Volume in Terms of a Double Integral
Exercise 8: Volume of the Standard 3-Simplex

Document: 12.3 The Gamma and Beta Functions

Movie: 12.3 The Gamma and Beta Functions

12.3.1 The Equation $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$

The Equation $\lim_{x\to\infty} \frac{x^0}{e^x} = 0$
The Equation $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$ When p Is Negative
The Equation $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$ When p Is Positive

12.3.2 Introducing the Gamma Function

Definition of the Gamma Function
Some Examples to Illustrate the Gamma Function
A Harder Example
The Convergence of the Integral $\int_0^\infty x^{a-1}e^{-x}dx$
The Graph of the Gamma Function

12.3.3 Some Elementary Facts About the Gamma Function

The Recurrence Formula
The Gamma Function and Factorials
The Substitution $x = t^2$
The Value of $\Gamma\left(\frac{1}{2}\right)$

12.3.4 Introducing the Beta Function

Definition of the Beta Function
Some Examples to Illustrate the Beta Function
The Convergence of the Integral $\int_0^1 t^{a-1}(1-t)^{b-1}dt$
The Graph of the Beta Function

12.3.5 Some Elementary Facts About the Beta Function

Symmetry of the Beta Function
The Substitution $u = ct$
The Substitution $t = \sin^2\theta$
The Value of $B(\frac{1}{2}, \frac{1}{2})$

12.3.6 The Relationship Between the Gamma and Beta Functions

Introducing the Relationship
Proof of the Formula $\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$

12.3.7 Some Exercises on the Gamma and Beta Functions

Exercise 1: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
Exercise 2: $\Gamma\left(\frac{13}{2}\right)$

Exercise 3: $\int_0^{\pi/2} \cos^8\theta \sin^{12}\theta d\theta$

Exercise 4: $\int_0^{\pi/2} \cos^7\theta \sin^{12}\theta d\theta$

Exercise 5: $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Exercise 6: $\int_0^{1} \sqrt{1 - x^4}\, dx$

Exercise 7: $\int_0^{1} \frac{1}{\sqrt{1-x^4}} dx$

Exercise 8: $\int_0^{\infty} \frac{1}{\sqrt{1+x^4}} dx$

Exercise 9: $\iint_{Q^2} x^{p-1} y^{q-1} dxdy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)}$

Exercise 10: $\int_0^{\pi/2} \sin^p\theta d\theta = \int_{\pi/2}^{\pi} \sin^p\theta d\theta$

Exercise 11: $B(a,a) = \frac{1}{2^{2a-1}} B\left(a, \frac{1}{2}\right)$

Exercise 12: $\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a)\Gamma\left(a + \frac{1}{2}\right)$

Exercise 13: $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$

Exercise 14: $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

12.3.8 A Hard Fact About the Gamma Function

Statement of the Hard Fact
An Application of the Hard Fact

Document: 12.4 Changing Integrals to Polar Coordinates

Movie: 12.4 Changing Integrals to Polar Coordinates

12.4.1 Introducing the Change to Polar Coordinates

A First Look Changing to Polar Coordinates
A More Careful Description of the Regions of Integration
Motivating the Formula for Changing to Polar Coordinates

12.4.2 Exercises on Polar Coordinates

Exercise 1: Using Polars to Evaluate an Integral
Exercise 2: Using Polars to Evaluate an Integral
Exercise 3: Using Polars to Evaluate an Integral
Exercise 4: Using Polars to Evaluate an Integral
Exercise 5: Using Polars to Evaluate an Integral
Exercise 6: Using Polars to Evaluate an Integral
Exercise 7: Using Polars to Evaluate an Integral
Exercise 8: Using Polars to Evaluate an Integral
Exercise 9: Using Polars to Evaluate an Integral
Exercise 10: Using Polars to Evaluate an Integral
Exercise 11: Using Polars to Evaluate an Integral
Exercise 12: Using Polars to Evaluate an Integral
Exercise 13: Using Polars to Evaluate an Integral
Exercise 14: Using Polars to Evaluate an Integral

Document: 12.5 Integration of a Function of Three Variables

Movie: 12.5 Integration of a Function of Three Variables

12.5.1 Iterated Integrals in Three Variables

Iterated Integrals with Constant Limits
More General Iterated Integrals

12.7.13 Some Examples of Parametric Regions in $\boldsymbol{R}^3$

Example 1
Example 2
Example 3
Example 4

12.7.14 Integrating on a Solid Parametric Region in $\boldsymbol{R}^3$

Definition of the Integral of a Function on a Solid Parametric Region

12.7.15 The Boundary of a Solid Parametric Region in $\boldsymbol{R}^3$

The Boundary of the Standard 3-Simplex Q^3
The Boundary of a Rectangular Box in $\boldsymbol{R}^3$
Defining The Boundary of a Solid Parametric Region in $\boldsymbol{R}^3$
The Boundary of the Unit Ball in $\boldsymbol{R}^3$

12.7.16 A Change of Variable Formula for Integrals on the Boundary of a Solid Parametric Region

Introduction to the Change of Variable Theorem
A Needed Tool from Linear Algebra
Proving the Change of Variable Formula

12.7.17 The Gauss Divergence Theorem

Introduction to the Gauss Divergence Theorem
The Divergence Theorem on the Standard 3-Simplex Q^3
The Divergence Theorem on a rectangular box
The Gauss Divergence Theorem for Parametric Regions
Proof of the Gauss Divergence Theorem for Parametric Regions
The Gauss Divergence Theorem for Triple Integrals
Proof of the Gauss Divergence Theorem for Triple Integrals

12.7.18 Examples to Illustrate the Gauss Divergence Theorem

Exàmple 1
Exàmple 2
Exàmple 3